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Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin – I

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Abstract

We discuss supersymmetric Yang-Mills theories with the multiple scales in the brane language. The issue concerns $N = 2$ SUSY gauge theories with massive fundamental matter including the UV finite case of $n_f = 2n_c$, theories involving products of $SU(n)$ gauge groups with bifundamental matter, and the systems with several parameters similar to Λ_{QCD}. We argue that the proper integrable systems are, accordingly, twisted XXX $SL(2)$ spin chain, $SL(p)$ magnets and degenerations of the spin Calogero system. The issue of symmetries underlying integrable systems is addressed. Relations with the monopole systems are specially discussed. Brane pictures behind all these integrable structures in the IIB and M theory are suggested. We argue that degrees of freedom in integrable systems are related to KK excitations in M theory or D-particles in the IIA string theory, which substitute the infinite number of instantons in the field theory. This implies the presence of more BPS states in the low-energy sector.

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1 Introduction

During the last years it has become clear that the low energy sector of $N = 2$ supersymmetric gauge theories [1, 2] is governed by integrable systems with the finite number of degrees of freedom so that physical quantities in the gauge theories (couplings and masses) turn out to be parameters of these finite-dimensional systems. There have been considered several examples that include the pure gauge theory [3] as well as the theories with adjoint [4] and fundamental matter [5]. In particular, in [5] the $N = 2$ super-QCD with the number of flavors $n_f < 2n$ and the gauge group $SU(n)$ [5] has been considered as a first example of the theory with several scales treated in the integrability framework. In the present paper, we discuss more examples of the asymptotically free theories with the multiple scales. This condition implies the set of possibilities. Indeed, one can consider the theory with massive fundamental matter, the theory with many Λ_{QCD} type parameters instead of the single one, and the system with the gauge group being product of simple factors [6] so that there is a parameter attached to each factor. We find integrable counterparts to all these examples.

In fact, there are some (apparently different) models among field theories, supersymmetric monopoles, string theory compactifications which possess the same integrable structure that naturally adds to this list another integrable system counterpart. The whole picture is drawn in Fig.1.

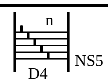
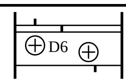
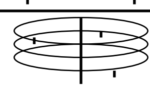
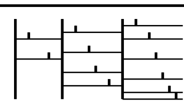
Integrable System	SUSY Field Theory	Brane Picture	Monopoles	Geometrical Engineering
(n-periodic) Toda Chain	Four-dimensional SU(n) Yang-Mills			A_{n-1}-type singularity
XXX Sl(2) Spin Chain	d=4 SU(n) QCD			A_m chain of spheres over "gauge" P^1
Calogero-Moser Model	YM theory with adjoint matter			
Inhomogeneous Sl(k) magnet	$\prod_{i=1}^{k-1}$ SU(n_i) theory		k SU(n) Monopoles	A_n-chain of spheres over each sphere in A_k-chain
XXZ / Ruijsenaars Model	d=5 theory			M theory over CY threefold
XYZ Spin Chain	d=6 theory			

Figure 1: Various aspects (columns) of the same models (rows)

Let us stress the special role of quiver diagrams in all these models. The same diagram defines also the corresponding brane configurations and integrable systems. It also labels the type of singularity in Calabi-Yau manifold in the geometric engineering approach as well as S-duality group.

It is clear that the revealing of the hidden integrable structure in the vacuum sector of SYM theory can not be the aim by itself, thus, we have to explain the whole motivation of the approach. First, let us note that the appearance of the integrable system can be expected from the very beginning. Indeed, integrable system typically has its configuration or phase spaces realized as some moduli space. That is what we have in the super-Yang-Mills (SYM) case: in the pure $N = 2$ gauge theory, the Coulomb moduli space and instanton moduli space are involved. Therefore, we would actually expect a pair of integrable systems. This is, indeed, the case, and the two integrable systems – the Whitham and Hitchin-like ones are exhibited. Identification of the Whitham dynamics as the one on the Coulomb moduli is quite transparent but the direct relation of the Hitchin dynamics with the instanton moduli space is still to be clarified. It has been shown that it is related to the monopole rather than to instanton dynamics. Still it is easy to connect these two – since the monopole

solutions can be constructed via an infinite dimensional counterpart of the instantonic ADHM construction [7], one could naturally identify the instantons taken in the conformal t'Hooft anzatz and organized into n_c infinite chains with the circle of n_c monopoles [8]. In what follows we use a kind of Born-Oppenheimer approximation and freeze the "slow" Whitham dynamics considering only the "fast" Hitchin one.

The monopoles which is one of the main subjects of the paper[1] have a specific nature – their role is to collect all the instanton nonperturbative contributions in $4d$ theory. That is why the brane configurations we discuss necessarily include the objects which provide the nonperturbative (instanton) effects in the field theory so that, as a result, D0-branes within D4-brane gives rise to the nonperturbative solutions of the monopole type. Note that we add this ingredient to the M theory approach in comparison with [6], which corresponds to additional KK degrees of freedom on the M5 worldvolume. This is in agreement with the general description of the finite-gap solutions of integrable systems which can be formulated as dynamics of points on the spectral curve. Within this approach the Lax equation is responsible for the structure of the fermionic zero modes in the monopole background, while the Baker-Akhiezer function would be associated with the properly normalized fermionic zero mode itself. The role of the spectral curve is to capture the band structure of the zero modes which are delocalized in the topologically nontrivial background. Note that just analogous band structure of the zero modes is expected in the usual QCD. Let us remark that there are similarities with the scenario of "confining strings" [9] which also are the effective objects which substitute the infinite number of instantons.

Turn now to the role which integrable systems play in the context of the low-energy SYM theories. First of all, the interpretation of the field-theory vacuum dynamics in terms of finite number degrees of freedom immediately raises the question, what these degrees of freedom really are. Now they are recognized as reflecting the dynamics of the finite number of interacting branes. Second, they provide the powerful computational tools to deal with the vacuum moduli spaces. Although the D-brane approach looks more transparent, it actually reflects the degrees of freedom of integrable systems and, therefore, the quantitative analysis available in the dynamical system seems to be very useful. For instance, the phase spaces of integrable systems always have an algebraic nature, thus, one usually has a huge underlying symmetry group for the corresponding dynamics [10]. Typically, they are of Yangian or quantum affine type. It is natural to look for the counterparts of these symmetries in the brane language. These algebras actually gives a field theory analog of the string "BPS like" algebras. We comment this subject in section 8. Let us remark that, amazingly, this point of view sometimes can provide even with a more convenient physical insight [11]. Another important point about the integrable treatment is that the finite dimensional integrable systems admit the Lagrangian description that can be considered as the starting point for the quantization of the D-brane transverse degrees of freedom.

Let us note that in this paper we intensively use the term "multiscale" that we attribute to the three different cases. First, we consider the $N = 2$ theory with fundamental matter where scales are introduced by the quark masses. The second example deals with the "group product" case with the distance between NS5 branes taken as dimensional parameters[2]. And, finally, the set of parameters can enter the low energy sector via the multiple regulator parameters due to a proper dimensional transmutation procedure.

The paper is organized as follows. At first, we review the picture behind the theory with a single scale and discuss the relevance of the brane degrees of freedom for integrability. This is done in section 2. In section 3 we complete the identification of the integrable structure for the $SU(n)$ theory with fundamentals and show that the theory with $n_f = 2n$ is governed by the twisted XXX $SL(2)$ spin chain. Section 4 is devoted to the clarification of the role of higher magnets as the integrable counterparts for the "group product" family of theories. In the course of our investigation, we works mainly with the "lattice" Lax representations [12] that can be characterized by the quadratic Poisson brackets with numerical r-matrix. Still, to deal with the

[1] Fix right here that, through the paper, speaking of monopoles, we often imply just a solution to the Nahm equations without specializing its boundary conditions. These Nahm equations is the result of applying the ADHM construction, and, to describe the true monopole, they are to be added by the proper boundary conditions. We return to the discussion of this point in s.6.

[2] These parameters can be associated with the finite cut-off in the UV finite theories or with the dimensional parameters of Λ_{QCD}-type in the asymptotically free ones.

multiscale theories, we need "large" Lax representation of the spin Calogero type that is characterized by the linear Poisson brackets – maybe with dynamical r-matrix (this is, of course, nothing but Hitchin system type representation [13, 14, 15]). Quite astonishing, in many integrable systems one can meet these both Lax representations so that they lead to the same spectral curve. This fact, along with the different Lax representations and their interpretation in brane terms is contained in section 5. The fact that higher magnets emerge not accidentally gets more clear in section 6, where the correspondence between spin chains and monopoles is established. A new type of gauge theories with multiple Λ type parameters which is described by degenerations of the spin Calogero (Toda) system is introduced in Section 7. We also discuss there the corresponding brane interpretation. Some comments on the hidden symmetry groups are presented in section 8. In the Appendix, we demonstrate some very explicit formulas for the $SL(3)$ magnet that are aimed to illustrate some statements of sections 4 and 5 concerning the "group product" case.

2 Single scale theory

2.1 Generalities

We start with the general description of the single scale SUSY gauge theories. Hereafter SUSY YM theories are treated as the theories on the D-brane world-volumes. The starting point is the consideration of n parallel D-branes embedded in higher dimensions. In order to produce 4d theory, this world has 3+1 noncompact dimensions while the coordinates of D-branes in other dimensions are identified with the vacuum expectation values of the scalar components in the adjoint hypermultiplets of the theory on the world volume. In fact, some of these D-brane coordinates in the transverse directions provide the phase space for the integrable many-body problems.

On the field theory side we consider the softly broken $N=4$ SYM theory. This theory in the $N=1$ language has three adjoint hypermultiplets whose complex scalar components parametrize the remaining six dimensions. The potential energy of these scalars contains the term $V=\frac{1}{g^2}\mathrm{Tr}\sum_{i=1,2,3}[\phi_i,\phi_i^+]^2$ coming from the kinetic term and the contribution from the superpotential $W=\mathrm{Tr}\Phi_1[\Phi_2,\Phi_3]+M^2\mathrm{Tr}(\Phi_2^2+\Phi_3^2)$, where Φ_i are the $N=1$ adjoint superfields and M is the mass of the hypermultiplet. Minimization of the potential energy provides classical vacuum configurations that, in our case, correspond to the three scalar fields satisfying the nontrivial commutation relations. Note that the large mass term in the Lagrangian freezes the fluctuations transverse to the surface

$$\mathrm{Tr}(\Phi_2^2+\Phi_3^2)=const \tag{2.1}$$

which is an algebraic curve in the two-dimensional complex space.

In this section we will try to fix the degrees of freedom of the integrable systems in brane terms. Actually, we need to identify coordinates, momenta and coupling constants in integrable many-body systems. We will show that the momenta in the integrable system are related to the positions of D4-branes in the proper direction. These branes provide the worldvolume for the d=4 $N=2$ theory. Coordinates in the integrable system come from positions of other branes responsible for the nonperturbative effects [16], while the coupling constants correspond to the regulator masses (say, the mass of adjoint scalar in the Calogero case).

One more ingredient of the integrability to be explained concerns the meaning of the Lax equation and the Baker-Akhiezer function. Let us turn, for the sake of simplicity, to the Toda chain case. There are some arguments in favor of interpretation of the Baker-Akhiezer function as the fermion zero mode in a monopole background. Indeed, the eigenvalue problem acquires the form of the equation for the fermionic zero mode "Hamiltonian" and similar to the Peierls model treatment

$$c_i\Phi_{i+1}+c_{i-1}\Phi_{i-1}+p_i\Phi_i=\lambda\Phi_i \tag{2.2}$$

where $c_i=\exp(q_{i+1}-q_i)$ and boundary condition $\Phi_{i+n}(\lambda)=\exp(in\pi(\lambda))\Phi_i(\lambda)$ are imposed, $\pi(\lambda)$ being quasi-momentum. The elements of the Lax operator admit the interpretation as matrix elements of the spectral

parameter operator

$$
\begin{aligned}
c_i &= \int \Phi^+_{i+1} \lambda \Phi_i d\lambda \\
p_i &= \int \Phi^+_i \lambda \Phi_i d\lambda
\end{aligned}
\tag{2.3}
$$

The spectral curve with this interpretation plays the role of the dispersion law for the delocalized fermion zero modes.

There are two equivalent ways of constructing Seiberg-Witten solutions from string theory. The relation between them was discussed in [17, 18, 19]. The first one, called geometric engineering, deals with the Calabi-Yau threefold compactification in the vicinity of singularity [20, 17]. The other one is the field theory on the worldvolume of D4-branes stretched between parallel NS5-branes [6]:

$$
\left\{
\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
NS5 & + & + & + & + & + & + & - & - & - & - \\
D4 & + & + & + & + & - & - & + & - & - & - \\
D6 & + & + & + & + & - & - & - & + & + & +
\end{array}
\right.
\tag{2.4}
$$

where plus stands for extended direction of each brane and minus denotes transverse direction. Since fourbranes are finite in the x^6 direction, the long-range theory is a four-dimensional $N = 2$ gauge theory as we need. In the IIB case, the original supersymmetry (32 real supercharges) is broken down to $N = 2$ in $d = 4$ (8 real supercharges) via geometry of the Calabi-Yau manifold. D-branes, wrapped around shrinking cycles, correspond to BPS states that further break half of the supersymmetries. In the IIA case, the SUSY is broken by inserting branes into the (perturbatively) flat space[3]. The BPS states in this picture are membranes of M theory ending on a single NS5-brane. For the sake of unity that we are trying to reach in this paper, we consider both the Type IIA/M theory and the Type IIB theory in parallel.

2.2 Type IIB perspective

If one deals with the Type IIB/F theory, $4d$ $SU(n)$ theories without matter emerge as the world volume theories of the D7-branes wrapped around an elliptically fibered K3. In fact, the whole picture in this case is not so clear. Still some of the important ingredients can be observed. In particular, the key ingredient of the integrable system – Calogero model (see, e.g., [13] and notations therein) – namely, the bare spectral curve (torus) whose modulus is the bare coupling constant enters the picture automatically. Another crucial ingredient – holomorphic 1-form Φ that, after all, turns out to be the Lax operator corresponds to the D7-branes while a particular evaluation representation at the marked point is represented by the background branes. The n D7-branes are wrapped around the bare torus, therefore, their coordinates in the transverse directions are captured by the matrix Φ, if the torus is embedded into the two-dimensional complex surface as the supersymmetric cycle [21]. The degrees of freedom coming from the antiholomorphic form $\overline{A}$ (whose diagonal elements are the coordinates of the Calogero particles) can be attributed to the branes representing nonperturbative effects. Finally, the regulators enter the problem via the insertion of a Wilson line[4] in the proper representation at the fiber torus in K3. Eigenvalues of the Wilson line provide the scales in the theory. Proper treatment of the regulators is still incomplete but some identification in the orientifold terms [22] seems attractive.

Let us emphasize that the role of the 11-12 dimensions in F theory in this consideration is only to provide the necessary input of the symmetry structure, that is, affine algebra and its representation as well as the bare coupling constant of d=4 theory. Note also that, in this picture, the branes are wrapped around the bare not the actual spectral curve.

[3]For models without D6-branes.

[4]This Wilson line provides the spin variables in the spin Calogero model [14, 15] and reduces to just a coupling constant in the standard (spinless) Calogero model.

To provide some additional feeling about degrees of freedom of the integrable system let us consider the almost trivial "free" many-body integrable system. It corresponds to the d=5 theory without adjoint matter [23]. The convenient look at this system follows from the Chern-Simons (CS) interpretation of the Ruijsenaars system, now without interaction. The proper picture [24] can be presented in terms of the $SU(n)$ bundle over torus if one considers the CS theory on the product of this torus and an interval. The relation for monodromies around the cycles g_A, g_B reads

$$g_A g_B g_A^{-1} g_B^{-1} = 1$$

and the diagonal elements of matrices g_A, g_B which, in the free case, can be diagonalized simultaneously represent the coordinates and the momenta of the integrable system $g_A = diag(e^{ix_i}); g_B = diag(e^{ip_i})$. Remark that, from the F theory point of view, we have degenerated the fiber torus to a cylinder.

Let us compare this data with the brane picture. There are two coordinates in the brane picture relevant for the d=5 theory (let us denote them x^5, x^6), the typical size of the brane configuration in the x^5 direction being R_5^{-1} and that in the x^6 direction – $\frac{1}{g^2}$ [25]. We assume that the both coordinates are compactified onto the circles with radii above. Now our claim is that the tori in the Ruijsenaars model and in branes are related by the T duality transformation in the both directions. Therefore, the radii in the Ruijsenaars torus are $\alpha' R_5, \alpha' g^2$. According to the general rules, the eigenvalues of the monodromies around the cycles are transformed into the positions of n branes along the corresponding directions. To be precise, now we have n 5D branes of the IIB theory with the (worldvolume) x^6-positions which are localized at p_i in the x^5 direction and n D1 branes with x^5 worldvolume coordinates which are localized at x_i along x^6. It is assumed that our 4+1 theory is defined on first five coordinates of the worldvolume $(x^0, x^1, x^2, x^3, x^4, x^6)$ of n parallel D5 branes and positions of branes p_i in the x^5 direction correspond to the expectation value of one adjoint real scalar field on its worldvolume. Let us also note that the arising configuration is the simplest example of the "polymer picture" from [25] in the case when there are no "background" branes.

Let us now switch on the interaction in the many-body problem. This means that we add the massive adjoint hypermultiplet in the field theory approach, add the marked point with a nontrivial monodromy around in the CS picture, or add "background" branes in the string language. In the CS theory, the monodromy relation modifies to

$$g_A g_B g_A^{-1} g_B^{-1} = g_c$$

where the monodromy g_c corresponds to the $CP(n-1)$ coadjoint orbit at the marked point. After the T duality transformation, one gets the brane picture with a nontrivial distribution of the flux of the magnetic field on the torus surface. Actually, the flux distribution is dictated by the structure of the monodromy g_c, and one has equal fluxes M on all plaquettes except the ones on the "diagonal" δ_{ij}. This implies a constant magnetic field through the torus and n "vortexes" with the line structure on the diagonal which locally compensate the external magnetic field. Due to the external field the "brane polymer" is deformed and the local deformation on the i-th site $\delta p_i(x_{i+1} - x_i)$ is of order M in agreement with the Lax matrix structure. Note that the set of fluxes through the torus can be interpreted as the existence of background branes which are localized at a single point on a T dual torus.

2.3 Type IIA/M theory perspective

One of the possible deals with the Seiberg-Witten theory is to look for its origin in the M theory branes [6]. In order to get a four-dimensional gauge theory with eight real supercharges, i.e. $N = 2$ SUSY, we consider the field theory on the world volume of the M5-brane. Its worldvolume splits into the flat R^4 which is the space-time of the four-dimensional theory and a compact surface Σ of fixed genus g holomorphically embedded into the four-dimensional part Q_4 of the transverse space.

After compactification onto the M theory circle this single fivebrane looks as a set of Dirichlet fourbranes stretched between NS5-branes.

Let us briefly remind the basic ideas of the geometrical approach to the Seiberg-Witten theories [17,

20]. In order to get $N = 2$ four-dimensional $SU(n)$ gauge theory, one has to consider the Type IIA string compactification on the Calabi-Yau threefold near the A_{n-1} singularity. Blowing up the singularity results in changing the complex structure of the mirror IIB model:

$$xy = P_n(w) \tag{2.5}$$

If we have more than one singularity – that is, more than one gauge group, the appropriate picture involves an A_p-singularity on the base. Gradually blowing it up, we reduce its rank by one:

$$w^p + P_{n_1}(\lambda)w^{p-1} + xy = 0 \tag{2.6}$$

At the very end, one gets the "product of gauge groups":

$$w^p + P_{n_1}(\lambda)w^{p-1} + P_{n_2}(\lambda)w^{p-2}\cdots + xy = 0 \tag{2.7}$$

Incorporation of the fundamental matter means taking coupling of the $SU(n_f)$ group to zero, i.e. the size of the two-cycle to infinity. To have fundamental matter for each of the $SU(n)$ factors, replace the base P^1 by the chain of trivalent vertices, with fibers being the chains A_{n_i-1} of "gauge" P^1's and the chain A_{m_i-1} of matter P^1's over each "gauge" sphere.

It coincides with the Type IIA picture of Witten's construction [6] described above. To ensure they are the same theories, let us briefly revise the appearance of a fivebrane in geometrical approach. According to [26], the A_{n-1}-singularity on the fiber over P^1 in the Type IIB string theory is T-dual to the n coinciding fivebranes in the Type IIA theory. We can regard our ADE-type singularity as a singularity in the local model of $K3$. Then (2.5) in each root of $P_n(w)$ describes a degenerated torus. The monodromy $\tau \to \tau + 1$ around such singular fiber corresponds to $\rho \to \rho + 1$, i.e. $B \to B + 2\pi$ from the dual IIA point of view [26, 27]. In order to see that this region carries the unit H charge, integrate it over a three-cycle consisting of a circle S^1 around the point with degenerated fiber times the fiber T^2:

$$\frac{1}{2\pi}\int_{T^2\times S^1} H = \int_{T^2\times S^1}\frac{dB}{2\pi} = \frac{\Delta B}{2\pi}\mid_{S^1}= 1 \tag{2.8}$$

It has a straightforward generalization useful for the group product case that each singular point on the base corresponds to a $NS5$-brane in the brane language [17]. These singularities (fivebranes) have also clear interpretation in integrable models. Their counterparts are just marked points on the bare spectral curve. Although the elliptic models are not the subject of the present paper, they give an insight of this connection. Consider the class of Hitchin systems defined on torus as the bare spectral curve: (spin) Calogero model, (spin) Gaudin system, etc. Our conjecture is that all these models have a corresponding brane interpretation with marked points corresponding to the $NS5$-brane.

Another picture is actually formulated by Witten [6] (see also [27]) who suggested to treat all branes in the IIA configuration as projections of a single M5 brane wrapped around the full spectral curve (not bare one as in the IIB picture) via different sections.

Let us remark the only notion of the spectral curve is not sufficient to talk of integrability. Actually, we need an integrable dynamics that can be realized on a set of points (n for $GL(n)$ theory) moving on the spectral curve. In fact, it is right their dynamics that is integrable and is linearized on the Jacobian and just these degrees of freedom have been missed so far.

Thus, in order to get a complete integrable system we need the new ingredient – a set of points on the spectral curve which can be represented by the additional KK excitations. In $10d$ IIA picture they correspond to the nonperturbative corrections which are interpreted as D0 branes on D4 branes. But of course it is not a whole story – one has to introduce Λ_{QCD} via a set of "magnetic fluxes". The integrable system we are trying to get in the brane terms and which governs the answer in the $N = 2$ theory without matter is the periodic Toda chain. It is convenient to start with the theory with adjoint matter and send the mass of matter to infinity. Then, due to the dimensional transmutation phenomena a new scale – $\Lambda_{QCD} = M\exp(-\frac{n}{g^2(M)})$ emerges.

In the IIA brane picture, we have n D4 branes stretched between two NS5 branes. These D4 branes provide the worldvolume for d=4 theory. The $x^4 + ix^5$-coordinates of D4 perturbatively provide the momenta in the integrable system. There are also n D0 branes on D4 branes which would be ordered along the x^6 direction. In the Toda case, their coordinates can be naturally identified with

$$x_j = j\Delta + \phi_j \tag{2.9}$$

where Δ is a constant which defines the equilibrium position, ϕ_i represent the small fluctuations corresponding to the Toda coordinates and the variables x_i can be considered as the coordinates in the Calogero system (which has no any "lattice structure", that is to say, ordering). Note also that, by comparing to the field theory data, one gets $n\Delta = \frac{1}{g^2(M)}$ making our interpretation along x^6 direction natural. D0 on D4 behaves like a monopole, so we obtain something like the "monopole ring" which lies on the "diagonal".

The possible explanation of the local Toda Lax operator now looks as follows. The element L_{11} tells that the D0 is placed on the D4 brane so we can assign the same $x^4 + ix^5$ coordinate to it. Actually the whole issue of the Lax operators concerns the structure of the fermionic zero modes in the topologically nontrivial background, so the very possibility of 2×2 representation tells that we have no more then 2 fermionic zero modes here. The elements L_{12}, L_{21} correspond to the nontrivial matrix elements of the spectral parameter (2.3) between the fermionic zero modes localized on the neighbor D0 branes. At last, we would conjecture that vanishing of L_{22} reflects the fact that only one chiral zero mode exists for a monopole under consideration.

In what follows we exploit this monopole-related interpretation of the brane picture. Let us emphasize that we would fairly talk about the monopoles in the $SU(n)$ theory not about n monopoles of the $SU(2)$ one, the latter seems to appear because of some duality. Later we will confirm that just these $SU(n)$ monopole system can be immediately generalized in a more complicated situation. Let us also remark that evidently we have no the 2×2 monopole picture for the Calogero system which is in a perfect agreement with the absence of the second Lax representation (see section 5).

To complete the section outline once again that in any picture three main ingredients have to be included. First, one guesses the set of branes which have to provide the ultimate worldvolume for the d=4 theory with $SU(n)$ gauge group – these are n D7 in the IIB/F or M5 in the IIA/M pictures. Second, one has to add "instantons" which are M2 and KK excitations in the M theory and finally the regulator degrees of freedom are introduced – these are Wilson line on the torus in IIB/F or some "magnetic flux" analogues in M theory.

3 Simple gauge group and $SL(2)$ spin chains

In the section we discuss integrable systems that correspond to some $N = 2$ SUSY YM theories. Among these theories there are a few which are UV-finite. Integrable systems corresponding to the UV finite theories softly perturbed by mass terms serve as "reference systems" for us, since all other (asymptotically free) theories can be obtained from the UV-finite ones by some degenerations.

The defining property of integrable model behind UV-finite theory is that it is associated with elliptic "bare curve" whose modulus τ plays the role of the bare coupling constant in the theory. In some cases (theory with adjoint matter hypermultiplet [4]) this bare curve is an underlying Riemann surface where the Lax operator is defined. In other cases, considered in this paper, some coefficients (coupling constants) turn to be some modular forms of τ. In this latter case, one can ignore the bare curve in the four-dimensional consideration, just keeping in mind that it lives in the 11-12 dimensions of F theory and dealing with the coupling constants as just numbers (that are properly adjusted when doing the dimensional transmutation).

We start with the simplest case of the theory with the $SU(n)$ gauge group. The integrable structures corresponding to these case are well-known [3, 4, 5] and we reproduce it here merely in order to have some basic simple example.

This theory has zero β-function if one adds $2n$ massive matter hypermultiplets in the first fundamental representation. Before going into details let us note that in the brane picture [6] this case corresponds to

the two parallel NS5-branes with n D4-branes stretched between them and $2n$ additional D6-branes placed between NS5 that provides the UV-finiteness of the theory. In accordance with our general rules, it corresponds to the "reference" integrable system. This integrable system is the periodic inhomogeneous $SL(2)$ XXX spin chain with period n [5] has to be slightly twisted to make possible the dimensional transmutation to the asymptotically free theories. Let us note that, although throughout the paper we use the term "inhomogeneous $SL(p)$ magnet", it is equivalent, for the rational (XXX) case, to the homogeneous $GL(p)$ magnet.

3.1 UV-finite theory

The periodic inhomogeneous $sl(2)$ XXX chain of length n is given by the 2×2 Lax matrices

$$L_i(\lambda) = (\lambda + \lambda_i) \cdot \mathbf{1} + \sum_{a=1}^{3} S_{a,i} \cdot \sigma^a \tag{3.1}$$

σ^a being the standard Pauli matrices and λ_i being the chain inhomogeneities, and periodic boundary conditions. The linear problem in the spin chain has the following form

$$L_i(\lambda)\Psi_i(\lambda) = \Psi_{i+1}(\lambda) \tag{3.2}$$

where $\Psi_i(\lambda)$ is the two-component Baker-Akhiezer function. The periodic boundary conditions are easily formulated in terms of the Baker-Akhiezer function and read as

$$\Psi_{i+n}(\lambda) = -w\Psi_i(\lambda) \tag{3.3}$$

where w is a free parameter (diagonal matrix). One can introduce the transfer matrix shifting i to $i+n$

$$T(\lambda) \equiv L_n(\lambda) \dots L_1(\lambda) \tag{3.4}$$

which provides the spectral curve equation

$$\det(T(\lambda) + w \cdot \mathbf{1}) = 0 \tag{3.5}$$

and generates a complete set of integrals of motion.

Integrability of the spin chain follows from *quadratic* r-matrix relations (see, e.g. [12])

$$\left\{ L_i(\lambda) \overset{\otimes}{,} L_j(\lambda') \right\} = \delta_{ij} \left[r(\lambda - \lambda'),\ L_i(\lambda) \otimes L_i(\lambda') \right] \tag{3.6}$$

with the rational r-matrix

$$r(\lambda) = \frac{1}{\lambda} \sum_{a=1}^{3} \sigma^a \otimes \sigma^a \tag{3.7}$$

The crucial property of this relation is that it is multiplicative and any product like (3.4) satisfies the same relation

$$\left\{ T(\lambda) \overset{\otimes}{,} T(\lambda') \right\} = \left[r(\lambda - \lambda'),\ T(\lambda) \otimes T(\lambda') \right] \tag{3.8}$$

The Poisson brackets of the dynamical variables S_a, $a = 1, 2, 3$ (taking values in the algebra of functions) are implied by (3.6) and are just

$$\{S_a, S_b\} = -i\epsilon_{abc} S_c \tag{3.9}$$

i.e. $\{S_a\}$ plays the role of angular momentum ("classical spin") giving the name "spin-chains" to the whole class of systems. Algebra (3.9) has an obvious Casimir function (an invariant, which Poisson commutes with all the spins S_a),

$$K^2 = \mathbf{S}^2 = \sum_{a=1}^{3} S_a S_a \tag{3.10}$$

The spectral curve (3.5) can be presented in more explicit terms:

$$w^2 + \mathrm{Tr}T(\lambda)w + \det T(\lambda) = 0 \tag{3.11}$$

In accordance with (4.2), the last term defines masses of the hypermultiplets. Since

$$\det_{2\times 2} L_i(\lambda) = (\lambda+\lambda_i)^2 - K^2 \tag{3.12}$$

one gets

$$\begin{aligned}\det_{2\times 2} T(\lambda) = \prod_{i=1}^{n} \det_{2\times 2} L_i(\lambda) = \prod_{i=1}^{n}\left((\lambda+\lambda_i)^2 - K_i^2\right) = \\ = \prod_{i=1}^{n}(\lambda - m_i^+)(\lambda - m_i^-)\end{aligned} \tag{3.13}$$

where we assumed that the values of spin K can be different at different sites of the chain, and

$$m_i^{\pm} = -\lambda_i \pm K_i. \tag{3.14}$$

While the determinant of monodromy matrix (3.12) depends on dynamical variables only through Casimirs K_i of the Poisson algebra, the dependence of the trace $\mathrm{Tr}_{2\times 2} T(\lambda)$ is less trivial. Still, it depends on $S_a^{(i)}$ only through Hamiltonians of the spin chain (which are not Casimirs but Poisson-commute with *each other*) – see further details in [5].

Let us note that we have some additional freedom in the definition of the spin chain and the spectral curve. Namely, note that r-matrix (3.7) is proportional to the permutation operator $P(X \otimes Y) = Y \otimes X$. Therefore, it commutes with any matrix of the form $U \otimes U$. Thus, one can multiply Lax operator of the spin chain by arbitrary constant matrix without changing the commutation relations and conservation laws. Moreover, we can also insert a constant (external magnetic field) matrix V into the end of the chain (into the n-th site). This is the same as to consider more general boundary conditions – those with arbitrary matrix V^{-1}. This is why such a model is called twisted.

The described freedom allows one to fit easily the form of the spectral curve proposed in [2, 28]

$$\begin{gathered} w - \frac{Q(\lambda)}{w} = P(\lambda), \\ P(\lambda) = \prod_{i=1}^{n}(\lambda - \phi_i), \qquad Q(\lambda) = h(h+1)\prod_{j=1}^{2n}\left(\lambda - m_j - \frac{2h}{n}\sum_i m_i\right), \\ h(\tau) = \frac{\theta_2^4}{\theta_4^4 - \theta_2^4} \end{gathered} \tag{3.15}$$

where τ is the bare curve modular parameter and θ_i are the theta-constants [29].

It can be done, e.g., by choosing matrices U and V to be[5]

$$U_i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & h(h+1) \\ \prod_i \alpha_i & 0 \end{pmatrix} \tag{3.16}$$

i.e.

$$V^{-1} = \frac{1}{\det V}\begin{pmatrix} 0 & h(h+1) \\ \prod_i \alpha_i & -1 \end{pmatrix} \tag{3.17}$$

Let us remark that further extension of the theories with the fundamental matter would lead to the (twisted) XXZ and XYZ chains for the 5d and 6d theories correspondingly.

3.2 Pure gauge theory

Now let us briefly consider the degeneration of our reference system to the $n_f < 2n$ case. This can be done in the standard way [2] by taking l masses $m_1, ..., m_l$ to infinity while keeping $\Lambda_{QCD}^l \equiv e^{i\pi\tau} m_1 ... m_l$ finite. After this procedure the modular forms disappear so that Λ_{QCD} emerges instead.

[5] To fit (3.15), we also need to shift $\lambda_i \to \lambda_i - \frac{2h}{n}\sum_i m_i$.

Degenerations of the system can be studied at a single site (for the sake of brevity, we omit the index of the site). Let us consider the Lax operator $\widetilde{L} = UL$ (see (3.1), (3.16)) with spins satisfying the Poisson brackets (3.9). We are going to send α to zero. Along this line of reasoning, there are two possibilities to get nontrivial Lax operator. The first possibility, when the both masses (3.14) disappear and one reaches the pure gauge theory, is described by the periodic Toda chain [3]. In order to get it, one needs to redefine $S_+ \to \frac{1}{\alpha} S_+$, then take α to zero and remove, after this, the inhomogeneity by the shift of S_0. This brings us to the Lax operator of the form (we introduce new notations $S_0 = S_3$, $S_\pm = S_1 \pm i S_2$)

$$\begin{pmatrix} \lambda + S_0 & S_- \\ S_+ & 0 \end{pmatrix} \tag{3.18}$$

so that the Poisson brackets are

$$\{S_\pm, S_0\} = \pm S_\pm, \qquad \{S_+, S_-\} = 0 \tag{3.19}$$

This algebra is realized in new (Heisenberg) variables p and q

$$S_\pm = \pm e^{\pm q}, \qquad S_0 = p, \qquad \{p, q\} = 1 \tag{3.20}$$

This leads us finally to the Toda chain Lax operator [12]

$$L_{Toda} = \begin{pmatrix} \lambda + p & e^q \\ -e^{-q} & 0 \end{pmatrix} \tag{3.21}$$

and the spectral curve (3.11) takes the form

$$w^2 + \mathrm{Tr} T(\lambda) w + 1 = 0 \tag{3.22}$$

Now let us return to the second possibility of the asymmetric degeneration, when one of the masses (3.14) remains in the spectrum while the second one goes to infinity. One can understand from (3.14) that, in contrast to the Toda case, this degeneration requires a special fine tuning of the Casimir function and inhomogeneity, so that both of them go to infinity but their sum (difference) remains finite. In the Lax operator terms it can be done in the following way. Let us redefine $S_+ \to \frac{1}{\alpha} S_+$ *and* $S_0 \to \frac{1}{\alpha} S_0$. This means that the Poisson brackets take the form

$$\{S_+, S_-\} = -2S_0, \qquad \{S_\pm, S_0\} = 0 \tag{3.23}$$

Now in order to preserve finite Lax operator (3.1), one needs to take care of its element $L_{11}(\lambda)$. This can be done by the rescaling $\lambda_i \to \frac{1}{\alpha}\lambda_i$ and fixing $\lambda_i + S_0$ to be $\alpha \cdot s_0$. This brings us to the Lax operator

$$L(\lambda) = \begin{pmatrix} \lambda + s_0 & S_- \\ S_+ & \lambda_i - S_0 \end{pmatrix} = \begin{pmatrix} \lambda + s_0 & S_- \\ S_+ & -2S_0 \end{pmatrix} \tag{3.24}$$

The determinant of this Lax operator is equal to $(\lambda - m)$ where m is the finite mass

$$m = 2s_0 S_0 + S_+ S_- \tag{3.25}$$

in perfect agreement with (3.14), Let us note that $2s_0 S_0 + S_+ S_-$ is also the Casimir function of the algebra (3.23), since $\{S_\pm, s_0\} = \pm S_\pm$.

In fact, the Lax operator (3.24) can be rewritten in the form

$$L(\lambda) = \begin{pmatrix} \lambda + s_+ s_- - m & s_- \\ s_+ & 1 \end{pmatrix} \tag{3.26}$$

where we rescaled the spins and used the explicit form of fixed the Casimir function $2s_0 S_0 + S_+ S_-$. This Lax operator defines so-called novel hierarchy [30] and is gauge equivalent to the discrete AKNS [31] and relativistic Toda chain [32]. This latter correspondence looks especially unexpected, since the same relativistic Toda chain describes the 5d pure gauge SUSY theories [23].

Let us note that another (equivalent) way to count all possible degenerations [33] is to consider 2×2 Lax operator of the most general form linear in λ, which satisfies the Poisson brackets (3.6) with the rational r-matrix (3.7) and determine all Casimir functions with respect to this Poisson brackets. Then, all possible degenerations are in one to one correspondence with the vanishing of Casimirs[6].

To complete this section, let us say a few words of the brane interpretation of the Toda variables p_i. First, restore the dependence on Λ_{QCD} in the Toda Lax operator. It can be done in many different ways, we choose it to be

$$L_{Toda} = \begin{pmatrix} \lambda + p & \Lambda_{QCD}^2 e^q \\ -e^{-q} & 0 \end{pmatrix} \tag{3.27}$$

thus the spectral curve is of the form

$$w^2 + \mathrm{Tr}T(\lambda) + \Lambda_{QCD}^{2n} = 0 \tag{3.28}$$

Now taking the perturbative limit, i.e. putting $\Lambda_{QCD} \to 0$, we get that the spectral curve degenerates to the two sphere touching each other in the n points that are zeroes of the polynomial $\mathrm{Tr}T(\lambda)$. These zeroes are equal to p_i in the perturbative limit and describe the positions of the D4-brane ends in the $(x^4 + ix^5)$-plane, in accordance with the picture of [6].

There is another class of the spin chain variables, namely, those describing masses of the sixbranes (3.14), which can also allows very rough but transparent brane interpretation. Indeed, in accordance with (3.14), one can associate with each D4-brane (i.e. site of the spin chain) a pair of the sixbranes, the distance between them being measured by the value of the Casimir function K. At the same time, the $(x^4 + ix^5)$-coordinate of the center of masses of these two sixbranes is naturally measured by the inhomogeneity λ_i. This looks especially reasonable, since just the inhomogeneity corresponds to the decoupling $U(1)$ factor of the $GL(2)$ magnet.[7]

Note that, within this interpretation, the procedure of the degeneration gets its pictorial explanation. It just corresponds to removing one of the sixbranes to infinity. To keep the other branes at finite distances, one needs to tune parameters λ_i and K so that one of the two quantities $-\lambda_i \pm K$ remains finite.

4 The group product case

4.1 Reference system

Consider now more complicated case of the gauge group which is the product of some $SU(n)$ factors: $G = \prod_{i=1}^{p-1} SU(n_i)$. One can also add $p-2$ massive hypermultiplets in the representations $(n_i, \overline{n}_{i+1})$ and also by some hypermultiplets in the first fundamental representations. This case was first considered by E.Witten in [6] and possesses a new important feature. To start with, let us note that in this case for many different choices of n_i the suitable number of matter hypermultiplets can be added to make β-function zero. Actually, one needs to add

$$d_i = 2n_i - n_{i-1} - n_{i+1}, \quad n_0 = n_p = 0 \tag{4.1}$$

[6]We are grateful to S.Kharchev for the useful discussions on this point.

[7]One more interpretation of (3.14) might look as follows. We can start with a free theory and would expect the monopoles in "the momentum space" by the following reason. For the free particle, there is the standard dispersion relation $E = \pm(\vec{p}^2 + m^2)^{\frac{1}{2}}$ which tells us that there is the level crossing on the multi-dimensional surface $\vec{p} = \pm im$ in the complex momenta space. Would be this surface one-dimensional, according to the general philosophy of the Berry phase, this means that one needs to introduce two monopoles in "the momentum space" with their centers located at points $p = \pm im$. This resembles formula (3.14) without inhomogeneities. Furthermore, one can switch on the interaction with the Higgs field that results into the shift of the fermion mass due to the Yukawa coupling. Since inhomogeneities are naturally related to the Higgs vev's, one finally arrives at formula (3.14).

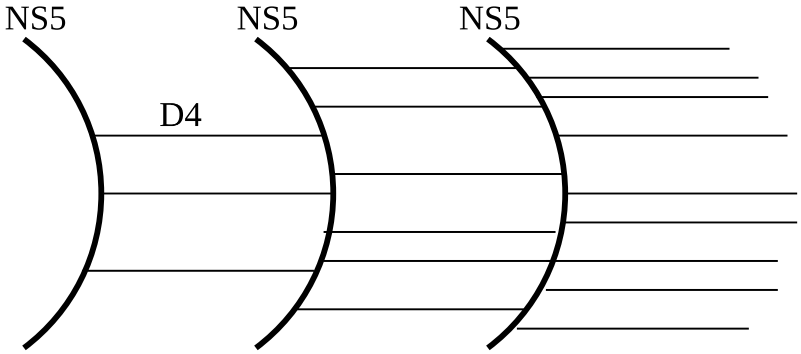


Figure 2: Brane picture, corresponding to the $Sl(3)$ magnet

hypermultiplets [6]. This always can be done unless some $n_i - n_{i+1}$ are too large. However, among all the integrable systems describing these numerous UV-finite theories there is a unique one that plays the role of the reference system, since all others can be obtained by degenerations. This theory is specified by maximal possible number of the matter hypermultiplets at given p and n_1 and characterized by the set $\{n_i\} = n, 2n, 3n, \ldots, (p-1)n$. The corresponding number of the hypermultiplets is $n_1 + n_{p-1} = np$.

Now come to the brane picture. In this case we have p parallel NS5-branes with n_1 D4-branes stretched between the first pair of them, n_2 D4-branes stretched between the second pair of the NS5-branes etc, with the corresponding numbers d_i of D6-branes. The reference theory in this language corresponds to Fig.2.

Its degeneration to theories with fundamental matter (D6-branes) in brane languages corresponds that $\lambda = x^4 + ix^5$ positions of several fourbranes coincide. Namely, if k of n_{p-1} D4-branes have the same λ coordinates as $k-1$ of n_{p-2} and so on till the only D4-brane of n_{p-k} set, we can attach a fictitious D6-brane to the right infinite end of the D4-set with this value of $\lambda = x^4 + ix^5$, and move it through $k-1$ fivebranes to the left consequently annihilating the whole set of such D4's. This mechanism will be explained in details in section 5 via brane creation phenomenon.

In further assigning brane configurations with integrable systems, it will be of importance that, with any described branes system, there can be associated the spectral curve of the form [6]

$$\begin{gathered} w^p + g_{n_1}(\lambda)w^{p-1} + g_{n_2-d_1}(\lambda)J_1(\lambda)w^{p-2} + \ldots + \\ + g_{n_k-(k-1)d_1-(k-2)d_2-\ldots-d_{k-1}}(\lambda)J_1^{k-1}(\lambda)J_2^{k-2}(\lambda)\ldots J_{k-1}(\lambda)w^{p-k} + \ldots + \prod_k^{p-1} J_k^{p-k}(\lambda) = 0 \end{gathered} \tag{4.2}$$

Here $g_i(\lambda)$ is a polynomial of degree i depending on the Coulomb moduli[8]

and hypermultiplet masses, and

$$J_i = \prod_{j=d_{i-1}+1}^{d_i} (\lambda - m_j) \tag{4.3}$$

are pure mass terms (m_j's are masses of the hypermultiplets in the fundamental representations).

Let us note that this spectral curve gives another justification for our choice of reference system. Indeed, let us consider "the pure gauge case"[9] when all $J_i(\lambda) = 1$ [6]. Then, only the moduli space of the reference spectral curves contains the Z_n-symmetric curve, and, in this sense, is maximally possible complete (although

[8]Hereafter, we call "Coulomb moduli" the moduli corresponding to the vev's of the fields in the vector (gauge) multiplet.

[9]To simplify further terminology, hereafter we call the system that contains no fundamental hypermultiplets "the pure gauge theory", despite there are still presented $p-2$ hypermultiplets in the representations $(n_i, \overline{n}_{i+1})$. Our terminology refers to the fact of absence of any D6-branes and semi-infinite branes and takes its origin in the $SU(n)$ case.

the moduli space of curves that correspond to an integrable system is always not complete for sufficiently large genera and covers typically g-dimensional subspace of the complete $(3g-3)$-dimensional moduli space; the g moduli are just the Hamiltonians of integrable system and, therefore, the dimension of the configuration space is exactly g).

To clarify this point, let us introduce the new variable $Y = w - \frac{g_{n_1}}{p}$. Then, only with the choice $n_k = kn$ one can match the polynomials g_k so that the curve takes the form

$$Y^p = R(\lambda) \tag{4.4}$$

where $R(\lambda)$ is a polynomial of degree np, i.e. this curve is of genus

$$g = \left(\frac{np}{2} - 1\right)(p-1) \tag{4.5}$$

The same genus can be obtained immediately from the brane picture, if one implies that the D4-branes are blown up to the handles of the Riemann surface, while NS5-branes are the spheres that are glued by these handles. Then, n D4-branes between first two NS5-branes give $n-1$ wholes, $2n$ next branes give $2n-1$ holes etc. – totally, it is exactly $\left(\frac{np}{2}-1\right)(p-1)$ holes.

In fact, this interpretation of the Riemann surface (spectral curve) as the brane graph implied in [6] is justified by the calculation of genus of the general curve

$$w^p + g_{n_1}(\lambda)w^{p-1} + g_{n_2}(\lambda)w^{p-2} + ... + g_{n_k}(\lambda)w^{p-k} + ... + g_{n_p}(\lambda) = 0 \tag{4.6}$$

The degrees of polynomials in this expression are restricted by the conditions $n_1 \leq n_2 \leq ... \leq n_{p-1}$ and $n_1 \geq n_2 - n_1 \geq n_3 - n_2 \geq ... \geq n_p - n_{p-1}$. The second condition is the requirement for all the β-functions to be non-positive, i.e. for the theory to be asymptotically free or UV-finite. These conditions allow one to calculate easily the genus of the curve. For doing this, we use the elegant formula (see, for example, [34])[10] for the genus of the non-singular projective curve given in the affine map by the polynomial $F(w,\lambda) = \sum a_{ij}w^i\lambda^j$. The genus is merely equal to the number of the integer points in the Newton polygon Δ for F, i.e. the convex shell of the points $(i,j) \in \mathbf{Z}$ such that $a_{ij} \neq 0$ – see Fig.3.[11]

Since powers of w represent resolution of singularity on the base, and the degree n_i of each polynomial g_{n_i} corresponds to resolution of A_{n_i-1} singularity on the fiber, by construction, Δ is nothing but toric polyhedron responsible for the local geometry of Calabi-Yau threefold [17].

Following [17] each interior node of Δ define a compact divisor, thus generating a homology group. As it was explained in details in [27], Riemann surface Σ faithfully represents all the data of Calabi-Yau cycles, with correspondence between cycles and forms. In other words this means that each compact divisor counts an element of $H_1(\Sigma)$.

In the present paper, from the integrable system point of view, and in [17], from the geometrical side, it was independently inferred that to have UV-free four-dimensional theory, i.e. each $d_i \geq 2n_i - n_{i-1} - n_{i+1}$, means to consider only convex Δ.

Using the conditions onto the set of $\{n_i\}$ (the second condition just guarantees the convexity), one can easily get that the genus of the curve (4.6) is equal to

$$g = \sum_{i=1}^{p-1} n_i - p + 1 \tag{4.8}$$

[10] We are grateful to A.Chervov who has drawn our attention to this formula.

[11] This formula is proved by noting that, for each pair (i,j) inside the polygon, there is the holomorphic differential

$$d\omega_{ij} = \frac{w^{i-1}\lambda^{j-1}dw}{F'_\lambda(w,\lambda)} \tag{4.7}$$

.

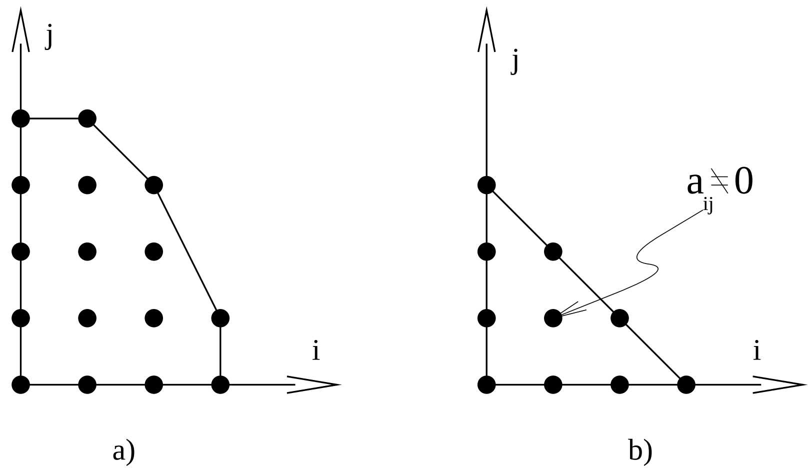


Figure 3: Newton polygon Δ

Let us note that this result coincides with one calculated from the brane picture as described above, since n_1 D4-branes between first pair of the NS5-branes gives $n_1 - 1$ holes etc. Remark also that in accordance with expectations, the degree of the last term does not enter the final answer, which means that the genus is not effected by the D6-branes (matter hypermultiplets) added.

Now, using the formula (4.8), we can determine the reference system as corresponding to the curve of the maximal possible genus at given p and $n_1 = n$.[12] This genus is equal, in accordance with (4.8), to $\left(\frac{np}{2} - 1\right)(p-1)$.

Below we describe the reference integrable system that is the periodic inhomogeneous $SL(p)$ (or homogeneous $GL(p)$) spin chain with period n. It is described by the curve (4.2) with all but J_{p-1} equal to 1.

4.2 UV-finite theory

The inhomogeneous $SL(p)$ spin chain is similar to the $SL(2)$ spin chain considered above and given by the $p \times p$ Lax operator

$$L_i(\lambda) = K^{ab} S_{a,i} X_a + (\lambda + \lambda_i) \cdot \mathbf{1} \tag{4.9}$$

at the i-th site. Here $S_{a,i}$ are dynamical variables[13], X_a are generators of the $SL(p)$ algebra and K^{ab} is its Killing form. We always choose below X_a to lie in the first fundamental representation of $SL(p)$. In complete analogy with (3.2), this Lax operator shifts the Baker-Akhiezer function to the neighbor site:

$$L_i(\lambda)\Psi_i(\lambda) = \Psi_{i+1}(\lambda) \tag{4.10}$$

and periodic boundary conditions read as

$$\Psi_{i+n}(\lambda) = -w\Psi_i(\lambda) \tag{4.11}$$

Introducing the transfer matrix shifting i to $i+n$

$$T(\lambda) \equiv L_n(\lambda) \dots L_1(\lambda) \tag{4.12}$$

and using boundary conditions, one gets the spectral curve equation

$$\det(T(\lambda) + w \cdot \mathbf{1}) = 0 \tag{4.13}$$

[12]Note that, by the shift of w, $g_{n_1}(\lambda)$ can be cancelled. For the reference curve, however, this shift does not change the degrees of all the rest polynomials $g_{n_i}(\lambda)$.

[13]To reproduce the $SL(2)$ case considered above, one needs to make the replacement $S_0 \to 2S_0$.

In fact, (4.13) is a polynomial in w and λ with the coefficients that give an ample set of integrals of motion.

The Poisson structure is in charge of integrability of the system and is given by (3.6) with the rational r-matrix

$$r(\lambda)=\frac{K^{ab}X_a\otimes X_b}{\lambda}=\frac{P}{\lambda} \tag{4.14}$$

where P is the exchange operator $P(X\otimes Y)=Y\otimes X$ and the last equality is correct, since we consider the fundamental representation. This gives rise to the Poisson brackets of S_i

$$\{S_a,S_b\}=-f_{ab}^{c}S_c \tag{4.15}$$

where f_{ab}^{c} are the structure constants of the $SL(p)$ algebra. The Poisson structure is, generally speaking, degenerated so that its annulator is the algebra of Casimir functions that are defined to be invariant with respect to the coadjoint action. Therefore, one needs to restrict the Poisson bracket to the orbits of this action.

Traces of degrees of Lax operators are Hamiltonians which mutually commute with respect to this Poisson structure. It can be easily understood since (3.6) is multiplicative and, therefore, products of Lax operators satisfy the same Poisson brackets.

Now one can use the explicit formulas for the Lax operator to obtain the following spectral curve

$$w^p+g_1(\lambda)w^{p-1}+...+g_k(\lambda)w^{p-k}+...+g_p(\lambda)=0 \tag{4.16}$$

where g_k is a polynomial of degree kn. Manifestly, $g_1(\lambda)=\mathrm{Tr}T(\lambda)$, ..., $g_k=\sum_I\det M_I^{(k)}(\lambda)$, ..., $g_p(\lambda)=\det T(\lambda)$, where $M_I^{(k)}(\lambda)$ is the matrix obtained from the transfer matrix by removing k columns and k rows that intersect at the diagonal elements given by the multi-index I. This spectral curve is exactly (4.2) with all but J_{p-1} put equal to 1.

Apart from the spectral curve, there is another key ingredient of the integrable system which is of great importance in the supersymmetric theories. This is the meromorphic differential dS [3, 5, 35] that generates the spectrum of the BPS states in the theory [1, 2]. The defining property of dS is that its variations over Coulomb moduli gives holomorphic 1-forms. In the spin chains, this differential has the form [36]

$$dS=\lambda\frac{dw}{w} \tag{4.17}$$

Let us check that this is, indeed, the proper differential, i.e. its variations over the Coulomb moduli give the holomorphic differentials. Consider the curve (4.16) (or (4.6)) and write it in the form $F(w,\lambda)=\sum_{i,j=0}a_{ij}w^i\lambda^j=0$. The differential dS depends on moduli a_{ij} of the curve only through the dependence of λ determined by the equation $F(w,\lambda)=0$. Thus, we get

$$\frac{\partial dS}{\partial a_{ij}}=\frac{\partial\lambda}{\partial a_{ij}}\frac{dw}{w}=\frac{\partial F}{\partial a_{ij}}\frac{dw}{wF'_\lambda}=\frac{w^{i-1}\lambda^j dw}{F'_\lambda} \tag{4.18}$$

Now it remains to note that the coefficients a_{p-k,n_k} are just fixed numbers (unities)[14] that are not to be regarded as moduli, while $g_p(\lambda)=\prod(\lambda-m_i)$ does not contain the Coulomb moduli and only masses. Then, using (4.7) we finally get that the variations of dS over the Coulomb moduli do really generate all the holomorphic differentials.

In order to check that the inhomogeneous periodic $SL(p)$ spin chain is really the reference system we are looking for, one needs to check that it properly degenerates to reproduce the curve (4.2) with arbitrary $J_i(\lambda)$. Indeed, from (4.16) one reads off that $J_{p-1}=\det T(\lambda)=\prod_i^n\det L_i(\lambda)$. Each determinant $\det L_i(\lambda)$ is a polynomial of degree p with coefficients being Casimir functions. On the generic orbit, all zeroes of this polynomial and, therefore, masses are different. Thus, we obtain $J_{p-1}=\prod_i^n\prod_r^p(\lambda-m_{i,r})$, where $m_{i,r}$ refers to p different roots of $\det L_i(\lambda)$.

In order to get generic J_i, one needs to consider some coinciding masses *at the same site*. If only two masses coincide, one gets only $J_{p-1}\neq 1$ and $J_{p-2}\neq 1$. This case corresponds to a specific set of orbits with

[14] This property of (4.16) follows from the manifest form of the Lax operator (4.9).

one relation between Casimir functions. Since there are $p-1$ (= rank $SL(p)$) independent Casimir functions, there can exist maximally $p-1$ relation between them. The corresponding special orbits describe $J_1(\lambda)$. Certainly, at different sites, one can consider different degenerations, giving $g_p(\lambda)$ the general form as in (4.2).

Now this is nontrivial test of the whole construction to check that the spectral curve (4.16) with all these degenerations gets the form (4.2), i.e. as soon as two roots of the polynomial $g_p(\lambda)$ coincide, the polynomial $g_{p-1}(\lambda)$ gets simple zero at the same point; as soon as three roots of the polynomial $g_p(\lambda)$ coincide, the polynomial $g_{p-1}(\lambda)$ gets double zero at the same point, while the polynomial g_{p-2} gets simple zero at this point etc. This is, indeed, the case and in the Appendix we demonstrate this by the direct calculation in the simplest case of the $SL(3)$ spin chain.

To conclude this subsection, let us note that the interpretation of the mass formulas for the sixbranes proposed in sect.3 is naturally continued to the $SL(p)$ magnet. Now a site of the spin chain should be associated with one D4-brane suspended between the first pair of the NS5-branes, two D4-branes suspended between the second pair, ..., $p-1$ D4-branes suspended between the last pair. With this last set of the $p-1$ D4-branes is then associated p sixbranes, with their coordinates being described by the masses. In this situation, again, the inhomogeneity describes the (x^4+ix^5)-coordinate of the center of masses of the sixbranes, and bringing one of the branes to infinity describes the procedure of the degeneration.

4.3 Pure gauge theory

Now let us discuss decoupling of the fundamental hypermultiplets, that is to say, degeneration of the spin chain. As before, the degeneration can be manifestly done using the freedom of multiplying the Lax operator by any constant matrix, since the r-matrix (4.14) is again the permutation operator and commutes with the matrix $U\times U$. This procedure looks literally the same as in the $SL(2)$ case. Thus, let us now discuss only the maximally degenerated systems corresponding to the pure gauge theories. These are described by $SL(p)$-generalizations of the Toda chain system.

As in the $SL(2)$ case, it is sufficient to consider the Lax operator at a single site. In the $SL(p)$ case there are $p-1$ different possible degenerations of (4.9) generalizing Toda system and $p-1$ different Lax operators. They contain the spectral parameter λ in the only one diagonal term, in two diagonal terms etc. till $p-1$ diagonal terms and have unit determinant. Varying this different Lax operators along the chain, one can reproduce polynomials of different degrees in (4.2) with all $J_k(\lambda)=1$ (i.e. all $d_i=0$), the degrees being subject to the inequalities $n_1\le n_2\le\ldots\le n_{p-1}$ and $n_1\ge n_2-n_1\ge n_3-n_2\ge ...\ge n_p-n_{p-1}$. This really corresponds to the pure gauge theory, see [6]. These different possible polynomials correspond to several possible partitions of D4-branes between background D6-branes.

As we already showed in s.3.2 (very explicit formulas for the $SL(3)$ case can be found in Appendix), one can obtain the different degenerations just multiplying the Lax operator (4.9) by constant diagonal matrices U with 1, 2, ..., $p-1$ non-unit entries equal to α, rescaling some spin operators $S\to\frac{1}{\alpha}S$ and then taking α to zero. Here we demonstrate the equivalent way of doing briefly mentioned in the very end of s.3.2 [33].

Namely, let us start with the general $p\times p$ Lax operator that satisfies the Poisson brackets (3.6) with the rational r-matrix (4.14):

$$\{L_{ij}(\lambda),L_{kl}(\mu)\}=\frac{1}{\lambda-\mu}\left(L_{kj}(\lambda)L_{il}(\mu)-L_{il}(\lambda)L_{kj}(\mu)\right) \tag{4.19}$$

Consider specific representations that are described by the Lax operators of the form (we consider, for a moment, the homogeneous $GL(p)$ system instead of inhomogeneous $SL(p)$ one)

$$L_{ij}(\lambda)=a_i\lambda\cdot\mathbf{1}+A_{ij},\qquad \mathrm{Tr}A=0 \tag{4.20}$$

These Lax operators can be produced from (4.9) multiplication by a constant matrix. From (4.19) one can read off the Poisson brackets for the matrix A

$$\{A_{ij},A_{kl}\}=a_kA_{il}\delta_{kj}-a_iA_{kj}\delta_{il} \tag{4.21}$$

with a_i being Casimir functions. From these Poisson brackets one concludes that, for an element A_{ij} to be a Casimir function, there should be fulfilled the condition $a_i = a_j = 0$. Now let us look at the determinant of the Lax operator (4.20). As was explained above it has the form

$$\det L(\lambda) = a_1 a_2 ... a_p \lambda^p + C_2\lambda^{p-2} + C_3\lambda^{p-3} + ... + C_p \tag{4.22}$$

where C_i are the Casimir functions (this formula specifies their choice). In the general $SL(p)$ magnet, all $a_i \neq 0$ and can be put equal to 1. In this case, the determinant (4.22) is a general polynomial of degree p with p zeroes. Degenerating of this system implies that some (say, k) of these zeroes (masses of the hypermultiplets) are taken to infinity, i.e. some $a_i = 0$ and some (k) first Casimir functions vanishes too. The pure gauge theory corresponds to the limiting degeneration when the determinant (4.22) is equal to constant (unity). This is reached by putting some a_i and all Casimir functions but the last equal to zero. Let us note that there exist exactly as many as $p-1$ classes, since one can put 1, 2, ..., $p-1$ a_i's equal to zero. This completes the construction.

5 Two Lax representations

5.1 2×2 versus $n\times n$ representations

So far we dealt with "lattice" Lax representations [12], for the $SL(p)$ groups formulated in terms of $p\times p$ matrices whose product along the chain – transfer-matrix – gives rise to the spectral curve. However, it is often needed another type of the Lax representations given by $n\times n$ matrices that is adequate for the (spin) Calogero (Toda) systems (and, more generally, Hitchin systems [13, 14, 15]). In this case, the Lax operator itself generates the spectral curve, while no analog of the transfer-matrix is known.

However, it sometimes happens that the same integrable system can possess *both* Lax representations resulting in the same spectral curve and different but equivalent phase space descriptions for the system. In the brane language it can be formulated as the equivalence after the rotation of two brane configurations. An example of this situation will be considered in the next section. It appears that this issue can be formulated in a relatively rigorous way.

In this and partly the next sections, we discuss interrelations between different Lax representations and their interpretation in brane terms.

The simplest relevant example is the Toda chain which has the 2×2 Lax representation (see s.Toda)

$$L_{Toda} = \begin{pmatrix} \lambda + p & e^q \\ -e^{-q} & 0 \end{pmatrix} \tag{5.1}$$

with the linear problem (3.2) and the boundary conditions (3.3). This system can be reformulated in terms of $n\times n$ matrix. Indeed, consider the two-component Baker-Akhiezer function $\Psi_n = \begin{pmatrix} \psi_n \\ \chi_n \end{pmatrix}$. Then the linear problem (3.2) can be rewritten as

$$\psi_{n+1} - p_n\psi_n + e^{q_n - q_{n-1}}\psi_{n-1} = \lambda\psi_n, \qquad \chi_n = -e^{q_{n-1}}\psi_{n-1} \tag{5.2}$$

and, along with the periodic boundary conditions (3.3) reduces to the linear problem $\mathcal{L}(w)\Phi = \lambda\Phi$ for the $n\times n$ Lax operator

$$\mathcal{L}(w) = \begin{pmatrix} -p_1 & e^{\frac{1}{2}(q_2-q_1)} & 0 & & -\frac{1}{w}e^{\frac{1}{2}(q_n-q_1)} \\ e^{\frac{1}{2}(q_2-q_1)} & -p_2 & e^{\frac{1}{2}(q_3-q_2)} & \dots & 0 \\ 0 & e^{\frac{1}{2}(q_3-q_2)} & -p_3 & & 0 \\ & & \dots & & \\ -we^{\frac{1}{2}(q_n-q_1)} & 0 & 0 & & -p_n \end{pmatrix} \tag{5.3}$$

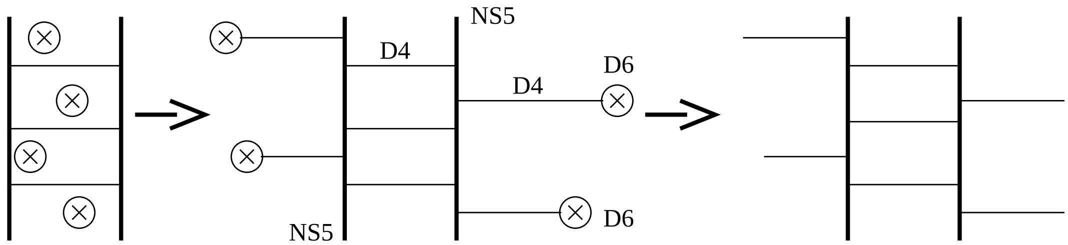


Figure 4: Different representations of fundamental matter first by inserting sixbranes, and then transformation of them into semi-infinite fourbranes

with the n-component Baker-Akhiezer function $\Phi = \{e^{-q_n/2}\psi_n\}$. This leads us to the spectral curve

$$\det_{n\times n}\left(\mathcal{L}(w) - \lambda\right) = 0 \tag{5.4}$$

which is still equivalent to the spectral curve (3.5).

The symmetry of two different representations of Lax operator by 2×2 or $N\times N$ matrices can be clearly reinterpreted in string theory language.

From the brane pictures point of view the two Lax representations are nothing but the choice of parametrization of the fivebrane worldvolume Σ. Of two holomorphic coordinates (which are Cartan elements of scalars) $\lambda = x^4 + ix^5$ and $s = x^6 + ix^{10}$, [15] we can regard any we wish as a spectral parameter. In the "standard" 2×2 representation, $w = \exp\frac{-s}{R}$ is a spectral parameter, and the spin chain is the appropriate integrable system. The corresponding typical brane picture is depicted in Fig.4. Another possibility is to choose λ as a spectral parameter. It leads to the $n\times n$ representation.

Geometrical image of this phenomenon is the base-fiber symmetry of the Calabi-Yau threefold [17]. As it was already mentioned earlier, in the "standard" 2×2 language w parametrizes the base A_m singularity and counts the number m of gauge factors. λ is a coordinate in the fiber. If each coefficient at w^i is a polynomial of degree n, we deal with the $SU(n)^m$ gauge theory with n extra fundamentals at each end of the chain of the gauge groups. The duality $SU(n)^m \leftrightarrow SU(m+1)^{n-1}$ proposed in [17] is nothing but the $m+1\times m+1 \leftrightarrow n\times n$ symmetry of the Lax representations. Under this symmetry, the Newton polygon (Fig.3) reflects so that axis i interchanges with axis j.

Note that the holomorphic differential

$$dS = \lambda\frac{dw}{w} = \lambda ds \approx -sd\lambda \tag{5.5}$$

also suggests this symmetry.

5.2 Sixbranes versus semi-infinite branes

Below we argue that the representation of the fundamental matter by the D6 branes and the semi-infinite D4 brane has different interpretation in terms of integrable systems. Namely, we would like to propose that the D6-branes are natural in the 2×2 representation, while the semi-infinite D4 branes in the $n\times n$ one.

We start with making use of the effect (first found by Hanany and Witten [37]) of creating a new brane when two other branes pass through each other.

In the paper, the authors considered the three dimensional theory in the common longitudinal directions of the branes:

[15] Equivalently, $s = x^6 + ix^9$ in the Type IIB language.

$$
\left\{
\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
NS5 & + & + & + & + & + & + & - & - & - & - \\
D3 & + & + & + & - & - & - & + & - & - & - \\
D5 & + & + & + & - & - & - & - & + & + & +
\end{array}
\right. \tag{5.6}
$$

By T-duality in the third direction, this setup gets mapped into Type IIA brane picture [6] described above (2.4) so that the statement of creating a new D3-brane when D5-brane passes through NS5-brane transforms into the following one: moving D6-brane through NS5-brane creates a new D4-brane stretched between them.

As it was also mentioned in [37, 6], the x^6-coordinate of the sixbranes apparently does not play any role, at least, in the low-energy physics we are interested in. Therefore, one can take these D6-branes to infinity, stretching a new D4-brane each time we pass through a NS5-brane. Moreover, we can move every particular sixbrane independently to the left or to the right up to our choice. The final configuration, shown in Fig.2, includes the original setup of fivebranes and fourbranes and very long fourbranes (with a sixbrane attached to the end of each D4-brane) instead of the sixbranes. Then, if the sixbranes (endpoints of the D4-branes) are quite far from the picture we are dealing with, we can replace this setup of branes by the same without any sixbranes, since they do not affect the physics. For example, the beta-functions of the gauge theories remain the same, since the presence of the D6-branes at infinity does not change the relative bending of the fivebranes. In other words, it means that we are going to the infinity of Q_4 which is Asymptotically Locally Flat.

The use of this phenomenon can be found in other columns of Fig.1. For instance, the existence of two different brane pictures which reveal the same low-energy physics suggests the existence of two different integrable counterparts. Really, the appropriate counterparts for the models with sixbranes are spin chains, while the semi-infinite fourbranes find their description in deformations of the Toda system by nontrivial boundary conditions suggested by I.Krichever and D.Phong [38]. Though at the low-energy level both systems are equivalent, the difference must appear in higher terms in the field theory, or higher times in the corresponding integrable systems. This also responds the question about the role of the x^6-coordinates of the sixbranes.

Let us more comment the same point from a slightly different view.

In the 2×2 representation, the spectral curve is (3.22). This spectral curve describes the pure gauge system obtained by degeneration of the (D6-branes + pure gauge) theory (spin magnet). Therefore, it is natural to identify the variable w in (3.22) with the variables y or z in [6]. These latter can be considered as describing one of the complex structures of the multi-Taub-NUT solutions

$$yz = \prod(\lambda - e_i) \tag{5.7}$$

where e_i are (x^4+ix^5)-coordinates of the sixbranes. It is of crucial importance that y and z never go to infinity at finite λ and, thus, describe the brane configuration that have only points at finite distances whenever λ is finite.

In contrast to the 2×2 representation, in the $n \times n$ representation the spectral curve has the form (5.4)

$$w + \frac{1}{w} = P(\lambda) \tag{5.8}$$

and is obtained from the Lax operator (5.3) given on a cylinder (double-punctured sphere). Let us note that the curves (3.22) and (5.8) are identical only if $w \neq 0$. If, however, one considers the projectivization of these spectral curves, they will become identical, with (3.22) and (5.8) corresponding to different maps. From the point of view of the Toda theory, this merely corresponds to different choices of gradation in Lax operator.

It is clear that the variable w lives on the cylinder with radius R which is identified with the radius of the compactified eleventh dimension in M theory [6]

$$w = t = e^{-\frac{x^6+ix^{10}}{R}} \tag{5.9}$$

The zero values of $w = t$ describe infinitely far points of the brane configuration. If now one adds the matter hypermultiplets in the fundamental representation, the spectral curve takes the form

$$w + \frac{Q(\lambda)}{w} = P(\lambda) \tag{5.10}$$

equivalent to (3.11) unless $w = 0$. However, there will be the points on the curve with zero values of $w = t$ (i.e. infinite values of physical coordinates) located at finite λ's. These points describe the semi-infinite D4-branes [6] and can be thought of as infinitely long handles, or punctures on the Riemann surface that correspond to the matter fundamental hypermultiplets in the standard picture [36].

The curve (5.10) can be immediately obtained from the $n \times n$ representation by the simple deformation of the Lax operator (5.3) just filling up the first and the last rows of the matrix (5.3) with any constant (independent on λ and proportional to w^{-1} and to w respectively) entries [38]. This Lax representation can be just obtained as already mentioned non-local boundary condition in the Toda chain [38]. Certainly, one can equally add non-zero entries to the first and the last columns. These two possibilities correspond to attaching the semi-infinite branes to the right or to the left NS5-branes respectively. We return to this construction again in the next subsection.

In a word, we would associate 2×2 spin chains with the system with matter fundamentals realized via inserting the sixbranes. On the contrary, the $n \times n$ representation is associated with the realization of the matter hypermultiplets via the semi-infinite branes.

There is another argument in favor of this identification that is related to the integrable system $\leftrightarrow$ brane configuration correspondence. Indeed, it is natural to consider the influence of the semi-infinite branes as a boundary effect like the emergence of $Q(\lambda)$ in the paper [38] is due to the non-local boundary conditions. At the same time, the sixbranes are more naturally associated with the "local" perturbations of the D4-branes suspended between NS5-branes, i.e. with additional degrees of freedom in the spin chain. This nicely fits the recent observation [39] that the semi-infinite D4-branes can be more naturally attributed to the meson not quark degrees of freedom.

5.3 $n \times n$ representation in the higher rank magnets

In this subsection we discuss a way to obtain $n \times n$ representations by degenerating the $SL(p)$ magnet. On this way, we reproduce the Lax operator of [38] that corresponds to bringing to infinity all but the very left two NS-branes. Then, the resulting system looks as the $SU(n)$ SYM theory with massive hypermultiplets realized via semi-infinite branes attached to the right NS-brane.

To begin with, let us note that, starting from eq.(5.4), one can maximally reproduce the curve (5.8) with polynomials $P(\lambda)$ of degree n and $Q(\lambda)$ of degree $2n - 2$. Thus, within the $n \times n$ representation, one can not reproduce the reference system (that requires the degree $2n$ of the polynomial $Q(\lambda)$) but can approach to it as close as possible[16].

Now let us consider the pure gauge limit of the $SL(p)$ magnet with the Lax operator with only one diagonal entry containing λ. From the results of s.4.3 we can conclude that it has the form

$$L_{p\times p}(\lambda) = \begin{pmatrix} \lambda + S_0^{(1)} & S_-^{(1)} & S_-^{(12)} & S_-^{(13)} & \dots & S_-^{(1,p-1)} \\ S_+^{(1)} & & & & & \\ S_+^{(12)} & & & & & \\ S_+^{(13)} & & & \widehat{A} & & \\ \vdots & & & & & \\ S_+^{(1,p-1)} & & & & & \end{pmatrix} \tag{5.11}$$

[16]Note that, following [38], one should consider $2n \times 2n$ Lax matrix to include the UV-finite case (i.e. to obtain $Q(\lambda)$ of degree $2n$) into the play. Thus doing, one is led to consider also any $pn \times pn$ Lax matrices – the result we can not interpret.

where $\widehat{A}$ is an ordinary constant $p \times p$ matrix of rank $p-2$ (thus, with zero determinant), $\det L_{p\times p} = 1$ and the Poisson brackets are

$$\left\{S_\pm^{(1i)}, S_0^{(1)}\right\} = \pm S_\pm^{(1i)}, \quad \left\{S_\pm^{(1i)}, S_\mp^{(1j)}\right\} = A_{j+1,i+1}, \quad i,j = 0,1,...,p-1, \quad S_\pm^{(1,0)} \equiv S_\pm^{(1)} \tag{5.12}$$

This Lax operator leads to the spectral curve

$$w^p + P_n^{(1)}(\lambda)w^{p-1} + \ldots + P_n^{(k)}(\lambda)w^{p-k} + \ldots + 1 = 0 \tag{5.13}$$

and, therefore, has no $n \times n$ representation (in accordance with the argument in the beginning of this subsection). However, one can restrict the matrix $\widehat{A}$ further so that the polynomial $P^{(k)}(\lambda)$ in (5.13) would be of degree $n-2k$. This system already has an $n \times n$ representation. It can be reached, for instance, in the case of $p = 3$ by the choice

$$\widehat{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{5.14}$$

in the case of $p = 4$ by the choice

$$\widehat{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{5.15}$$

etc. In the Appendix we consider how it happens in detail in the $p = 3$ case.

The Lax operator in this $n \times n$ representation has the following structure (for $n > p$):

$$\mathcal{L}_{n\times n}(w) = \begin{pmatrix} S_{0,1}^{(1)} & 1 & \ldots & & w^{-1} & \ldots & * \cdot w^{-1} \\ * & S_{0,2}^{(1)} & 1 & \ldots & w^{-1} & \ldots & * \cdot w^{-1} \\ * & * & S_{0,3}^{(1)} & 1 & \ldots & \ldots & * \cdot w^{-1} \\ & & & \ddots & & & \\ & & \ddots & & & & \\ & & & & & & \\ 0 & \ldots & * & \ldots & * & S_{0,p-1}^{(1)} & 1 \\ w & 0 & \ldots & * & \ldots & * & S_{0,p}^{(1)} \end{pmatrix} \tag{5.16}$$

so that the $p-1$ right corner diagonals are filled up by non-zero entries proportional to $\frac{1}{w}$ and $p-1$ diagonals below the main one are also non-zero. In formula (5.16) we use the asterisque to denote the non-zero entries which we do not write down manifestly for the sake of simplicity.

Here we use the normalization that slightly differs from (5.3). This Lax operator, after inserting into (5.4), indeed, leads to the spectral curve ($n > p$):

$$w^p + P_n^{(1)}(\lambda)w^{p-1} + P_{n-2}^{(2)}(\lambda)w^{p-2} + \ldots + 1 = 0 \tag{5.17}$$

Moreover, for $n = p$, one even gets the curve

$$w^p + P_n^{(1)}(\lambda)w^{p-1} + P_{n-1}^{(2)}(\lambda)w^{p-2} + \ldots + 1 = 0 \tag{5.18}$$

that is the maximal possible curve following from (5.4), at given p and n.

If now one brings all but the very left two NS-branes to infinity, the system looks as an $SU(n)$ system with matter hypermultiplets realized via the semi-infinite branes, the Lax operator (5.16) results into that of [38], and the spectral curve takes the form (5.8). Thus, the construction of [38], indeed, corresponds to the realization of the matter hypermultiplets via the semi-infinite branes.

5.4 Two Lax representations: general comments

Note that the two "dual" Lax representations has been presented in literature [40] in more invariant terms in the Gaudin limit of the Toda chain system. In the 2×2 representation, this limit corresponds to the

replacement $L_{2\times 2} \to 1+\epsilon L_G$ with small ϵ and leads to the *linear* Poisson brackets, instead of (3.6). Following [12], one can associate this replacement with transition to the continuum limit. In terms of the $n \times n$ Lax matrices, this limit corresponds to the rational limit of the trigonometric Toda r-matrix.

The relevant description of the Gaudin Lax representations can be presented via the embedding of this finite dimensional integrable system into rational coadjoint orbits of loop algebras. It is based on a family of the momentum maps from the space $M = \{F, G \in M^{n\times r}\}$ of pairs of $n \times r$ rectangular matrices, with a natural symplectic structure

$$\omega = \mathrm{tr}(dF \wedge dG^T) \tag{5.19}$$

to the dual of the positive half of the loop algebra $\widetilde{gl(r)}$. This algebra is defined as the semi-infinite formal loop algebra on $gl(r)$ that consists of elements $X(\lambda) = \sum_{i=-\infty}^{m} X_i\lambda^i$ with $X_i \in gl(r)$. Algebra $\widetilde{gl(r)}$ as the vector space has a natural decomposition

$$\widetilde{gl(r)} = \widetilde{gl(r)}^{+} \oplus \widetilde{gl(r)}^{-} \tag{5.20}$$

into the spaces of $r \times r$ matrix polynomials and strictly negative formal power seria in $\widetilde{gl(r)}$ respectively. We define the pairing between these two factors

$$\langle X(\lambda), Y(\lambda)\rangle = \mathrm{Tr}\left(X(\lambda)Y(\lambda)\right)_{-1} \tag{5.21}$$

where the subscript “-1” refers to the coefficient of λ^{-1}.

Now we fix an $n\times n$ matrix A whose spectrum is completely inside a disk D and define the group $\widetilde{GL(r)}^{+}$ of $GL(r)$-valued functions of λ that are holomorphic inside D.

Each this matrix A defines a symplectic action of $\widetilde{GL(r)}^{+}$ on M:

$$\begin{aligned} &g(\lambda):\ (F,G) \to (F_g, G_g)\,, \qquad g(\lambda) \in \widetilde{GL(r)}^{+} \\ &(A-\lambda)^{-1} F g^{-1}(\lambda) = (A-\lambda)^{-1} F_g + F_{hol}, \\ &g(\lambda) G^T (A-\lambda)^{-1} = G_g^T (A-\lambda)^{-1} + G_{hol}^T \end{aligned} \tag{5.22}$$

where F_{hol}, G_{hol}^T are holomorphic in D.

One can easily check using (5.21) [40] that this symplectic action is Hamiltonian, with moment map $J_A:\ M \to \left(\widetilde{gl(r)}^{+}\right)^{*}$ defined by

$$J_A(F,G) = -G^T (A-\lambda)^{-1} F \tag{5.23}$$

Absolutely analogously to this scheme, one can consider the loop algebra $\widetilde{gl(r)}^{+}$ and its group $\widetilde{GL(r)}^{+}$ and introduce a fixed $r\times r$ matrix $Y \in gl(r)$ that define the action

$$\begin{aligned} &h(w):\ (F,G) \to (F_h, G_h)\,, \qquad h(w) \in \widetilde{GL(n)}^{+} \\ &h(w) F (Y-w)^{-1} = F_h (Y-w)^{-1} + \overline{F}_{hol}, \\ &(Y-w)^{-1} G^T h^{-1} = (Y-w)^{-1} G_h^T + \overline{G}_{hol}^T \end{aligned} \tag{5.24}$$

where $\overline{F}_{hol}$, $\overline{G}_{hol}^T$ are holomorphic inside disk $\overline{D}$ containing the spectrum Y. As above, this action is Hamiltonian, with moment map $J_Y:\ M \to \left(\widetilde{gl(n)}^{+}\right)^{*}$ defined by

$$J_Y(F,G) = F (Y-w)^{-1} G^T \tag{5.25}$$

Now let us define two matrices

$$\begin{aligned} L(\lambda) &\equiv Y + G^T (A-\lambda)^{-1} F, \\ \mathcal{L}(w) &\equiv A + F (Y-w)^{-1} G^T \end{aligned} \tag{5.26}$$

Using the Adler-Kostant-Symes theorem, one now proves that these two matrices are Lax matrices [40]. Each of them is the Lax matrix of the classical Gaudin system. They can be also considered as solutions to the moment map equation of the Hitchin system on the sphere with n or, respectively, r marked points so that Y and A define the monodromy of infinity.

The equivalence of these two Lax representations is defined by the fact that they define bi-rationally equivalent spectral curves. This follows from the identity

$$\det(A-\lambda)\det\left(Y+G^T(A-\lambda)^{-1}F-w\right)=\det(Y-w)\det\left(A+F(Y-w)^{-1}G^T-\lambda\right) \tag{5.27}$$

Note that the construction for the Toda case looks more involved, since, in the 2×2 formalism, it is described by the quadratic Poisson brackets. This results in the product of the Lax operators along the chain (polynomial in λ of degree n). Actually we have to perform the transition from the algebra to the group level. Put it differently, this is the product over marked points, instead of sum that could be reproduced by the larger matrix still linear in the spectral parameter. Analogously, $n\times n$ Toda Lax operator has *linear* Poisson brackets with *trigonometric* r-matrix. This can be again reproduced by the product of two matrices (leading to a polynomial quadratic in w) instead of sum over the two marked points. Unfortunately, there is no technique developed for the products of matrices. It looks slightly similar to the averaging procedure proposed in [41], which is immediate for the linear Poisson brackets and is quite involved for the quadratic ones.

Let us also note that the natural analog of matrices F and G in the Toda case would be the $n\times 2$ matrix with the two columns made of $\{\psi_n\}$ and $\{\chi_n\}$.

Let us still say some words of the algebraic structures in the Gaudin system obtained from the Toda chain which corresponds to the above construction at $r=2$. In the $n\times n$ representation, the Lax operator is defined as the solution of the moment map equation for the holomorphic $SU(n)$ connection on the sphere with two marked points. The Lax operator in this representation contains momentum variables as the diagonal entries. This can be interpreted in the framework of the consideration above as the monodromy at infinity. Turn now to the 2×2 representation. Its structure clearly indicates that now momenta play the role of the positions of the marked points on the punctured sphere with the holomorphic coordinate λ that was the eigenvalue of the Lax operator in the first representation. The picture can be clearly generalized to the Gaudin system with arbitrary number of marked points in the first representation.

Returning to the brane interpretation behind the dual Lax representations, it can be viewed as follows. In the Type IIA, in one representation we "look from" the diagonal of the brane picture which gives rise to the 2×2 representation. It is this look that admits the interpretation in terms of an "effective n-monopole". In the other, $n\times n$ representation we take another look and obtain $SU(n)$ gauge group from the parallel D4-branes. The D0-branes provide the structure with a few monopoles in the $SU(n)$ theory leading to the $n\times n$ representation. In the IIB picture, we have $U(1)$ gauge field on NS5 brane and a possible interpretation of 2×2 representation is discussed in [37]. The other, $n\times n$ interpretation, in this case, comes from the wrapped branes around the compact surface.

6 Monopole $\leftrightarrow$ spin chain correspondence

It was already noted by different authors [42, 43, 37, 44] that monopole moduli spaces play an important role in the 4d SUSY gauge theories. The evident reason for the appearance of monopoles in this setup is the hyper-Kähler structure of the monopole moduli spaces which also can be manifested for the moduli space of vacua in SYM theories as well as for the phase space of the integrable systems under consideration. The useful approach to the description of the monopole moduli space is to introduce monopole spectral curve and look at the moduli space of such curves [45]. It was shown [7, 46] that the monopole spectral curve is related to the spectral curve of the so-called Nahm equations giving an infinite-dimensional counterpart of the ADHM construction [7, 46]. It was P.Sutcliffe who first used the Nahm equations in the context of the Seiberg-Witten

theories and noted that the solutions of the periodic Toda chain are naturally identified with the C_p cyclic $SU(2)$ monopoles of charge p.

It is worth noting that Nahm approach suggests an interpretation of the equations of motion of our integrable systems. Indeed, it was mentioned in [43] that the Nahm equations (and, therefore, the equations of motion of integrable system) are interpreted as the condition of the BPS saturation which keeps some supersymmetries. This means that the demand for the vacuum state in the proper σ model to be supersymmetric gives rise to the equations of motion. Note also that the fermions in the auxiliary Lax linear problem (Baker-Akhiezer function) would be related to the fermion zero modes in a monopole background.

In this section we demonstrate that such a correspondence between monopoles and integrable chains can be pushed further and state that the solutions of the spin chains can *exhaust all* the solutions to the Nahm equations. As a byproduct, we obtain a natural generalization of these equations. Physically the proposed correspondence implies a duality of some brane pictures. This can be understood in the framework of the Diaconescu construction [43], who generalized the analogous picture for instantons [47].

6.1 $SU(2)$-monopole versus spin chains

We start with the $SU(2)$ monopole[17] with charge p. It can be described by the Nahm equations [46] that are three equations for three $p \times p$ matrices $\mathcal{T}_i(t)$, $i = 1, 2, 3$

$$\frac{d\mathcal{T}_i}{dt} = \frac{1}{2}\epsilon_{ijk}\left[\mathcal{T}_j, \mathcal{T}_k\right] \tag{6.1}$$

given on the real segment $t \in [-1, 1]$ and regular on the interval $t \in (-1, 1)$. These matrices satisfy additional restrictions

$$\mathcal{T}_i(t) = -\mathcal{T}_i^{\dagger}(t), \quad \mathcal{T}_i(t) = \mathcal{T}_i^{\mathrm{tr}}(-t) \tag{6.2}$$

and boundary conditions

$$\mathcal{T}_i(t) \overset{t\to 1}{\sim} \frac{\mathcal{T}_i^{(1)}}{t-1}, \quad \mathcal{T}_i(t) \overset{t\to -1}{\sim} \frac{\mathcal{T}_i^{(-1)}}{t+1} \tag{6.3}$$

where $\mathcal{T}_i^{(1,-1)}$ form an irreducible p-dimensional representation of $SU(2)$. The numbers 1, -1 are just the chosen normalization of the asymptotics of the two Higgs field components at infinity. The physical monopole fields are constructed from the zero modes of the linear first order differential operator in the background field of the Nahm matrices $\mathcal{T}_i(t)$ [7, 48].

The three Nahm equations (6.1) can be rewritten as the single equation depending on the spectral parameter. For doing this, one needs to introduce the Lax operator

$$L_N(\lambda) = \mathcal{T}_+ + \mathcal{T}_0\lambda + \mathcal{T}_-\lambda^2 \tag{6.4}$$

where we denoted $\mathcal{T}_\pm = \mathcal{T}_1 \pm i\mathcal{T}_2$, $\mathcal{T}_0 = -2i\mathcal{T}_3$. Then, (6.1) can be written as the single Lax equation

$$\frac{dL_N(\lambda)}{dt} = \left[A(\lambda), L_N(\lambda)\right], \quad A(\lambda) = \frac{1}{2}\left(\mathcal{T}_+\lambda^{-1} - \mathcal{T}_-\lambda\right) \tag{6.5}$$

With the Lax representation, one can consider the isospectral problem and construct conservation laws from the spectral curve:

$$\det_{n\times n}(L_N(\lambda) + w) = 0 \tag{6.6}$$

This curve has the form [45, 46]

$$w^p + q_1(\lambda)w^{p-1} + ... + q_k(\lambda)w^{p-k} + ... + q_p(\lambda) = 0 \tag{6.7}$$

where $q_k(\lambda)$ is a polynomial of degree $2k$, and, therefore, the genus of the curve is $(p-1)^2$ (in accordance with (4.8). This curve is an algebraic curve in the space of the oriented geodesics in $\mathbf{R}^3$ isomorphic to the complex tangent bundle $T\mathbf{P}^1$ of the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ [45]. If x is the local coordinate on this sphere, (w, λ) are

[17] Hereafter, speaking of monopole we mean BPS monopole.

the standard local coordinates on $T\mathbf{P}^1$ given by $(w,\lambda) \to w\frac{\partial}{\partial x}|_{x=\lambda}$. Now conditions (6.2) and (6.3) can be rewritten in terms of the spectral curve S [45] as the reality condition

$$q_k(\lambda) = (-1)^k \lambda^{2k} \overline{q_k(-\lambda^{-1})} \tag{6.8}$$

and the following restrictions:

$$L^2 \text{ is trivial on the curve, } L(p-1) \text{ is real} \tag{6.9}$$

$$H^0(S, L^\omega(p-2)) = 0 \text{ for } \omega \in (-1,1) \tag{6.10}$$

where L^ω is the linear holomorphic bundle over $T\mathbf{P}^1$ given by the transition function $\exp(-\omega w/\lambda)$ defined on the domain $\lambda \neq 0, \infty$, and $L(p) = L \otimes \mathcal{O}(p)$.

In what follows we establish the "formal" correspondence between Nahm equations and spin chains ignoring the additional requirements (6.2), (6.3) or (6.8)-(6.10) that just restrict the admissible space of solutions. Thus, we concentrate on the general integrable properties of the Nahm system with no respect to the boundary conditions. In fact, these additional requirements are quite restrictive, since the monopole moduli space has the dimension $4p$ increasing with p linearly in contrast to the genus $(p-1)^2$ and the quadratic grow of the dimension of the moduli space of the general curves (6.7).

First of all, let us comment interpretation of Toda integrable equations as a particular Nahm equation (for the symmetric monopole configuration) [42, 44]. The key property here is that both coordinates and momenta of the integrable systems can be expressed in the monopole terms. More concretely, the Lax matrix of the integrable system $L(\lambda)$ can be written as a combination of the Nahm matrices for n symmetric $SU(2)$ monopoles (see below). However, first we see how the monopoles are getting involved in the picture. Actually as we discussed in section 2 we would like to relate the coordinates of Toda particles with the positions of the well separated monopoles in x^6-coordinate. The monopoles are localized on D4-branes, one per each brane, so we can attribute the same coordinate $p_i = x^{4+i5, D4i}$ to the monopole localized on the i-th D4 brane. Note that this is in agreement with the "inversion formula" of the Nahm construction [48]

$$\mathcal{T}_i^{km}(t) = \int d^3x \Psi^{+,k}(x,t) x_i \Psi^m(x,t) \tag{6.11}$$

which gives the Nahm matrices $\mathcal{T}_i(t)$ as the matrices of the coordinates in the Heisenberg picture in the space of the fermionic zero modes Ψ^k, $(k = 1,..,n)$ in the monopole background. The relevant Nahm matrices provide the momenta p_i in the Cartan sector, and coordinates e^{iq_n} for positive and negative roots

$$\mathcal{T}_1 = \frac{i}{2}\sum_{j=1} e^{iq_j}(E_{+j} + E_{-j}); \qquad \mathcal{T}_2 = -\sum_{j=1} e^{iq_j}(E_{+j} - E_{-j}); \qquad \mathcal{T}_3 = \frac{i}{2}\sum_j p_j H_j \tag{6.12}$$

where $E_{\pm j}$ are the generators corresponding to the simple and longest roots of the affine algebra $\widehat{SL(n)}$ and H_j are its Cartan generators. Note that the formulae above can be also interpreted in terms of the fundamental monopole of $SU(n)$.

Now let us return to the general monopole (Nahm) system and, first, let us note that the spectral curve (6.7) coincides with the spectral curve of the $SL(p)$ spin chain at two sites (4.16). In particular, one can easily get from the genus formula (4.5) for $n = 2$ the genus $(p-1)^2$ that is well-known answer for the monopole spectral curve [45]. Therefore, it is naturally to suggest that these two systems are equivalent. In fact, the identity of the spectral curves is still not sufficient to identify two systems. Thus, one needs to check the identity of the Lax representations. As soon as there is a gauge transformation that connects two Lax operators, the two systems are equivalent possessing the same integrals of motion. In this concrete situation should connect the Nahm Lax operator (6.4) and the Lax operator (4.9) of the $SL(p)$ spin chain. In fact, we assert that the 2-site transfer matrix

$$T_2(\lambda) = L_2(\lambda) L_1(\lambda) \tag{6.13}$$

reproduces (6.4). Indeed, the Nahm Lax operator is defined just as an arbitrary matrix polynomial quadratic in λ, i.e. all the three coefficients ($p \times p$ matrices) are arbitrary. On the other hand, the 2-site transfer matrix

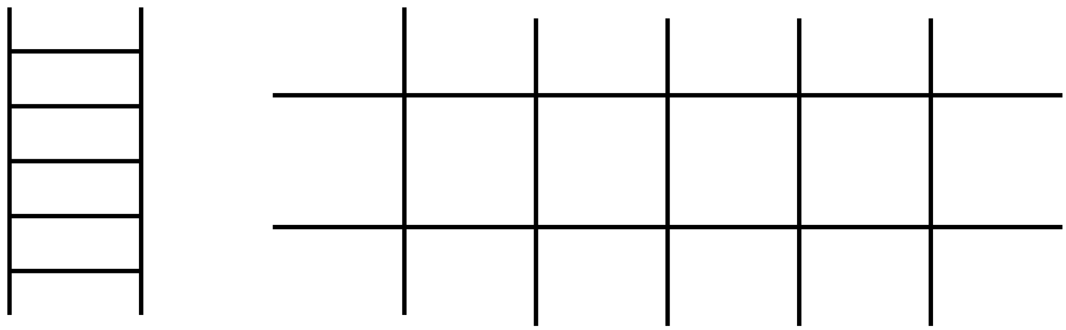


Figure 5: Two looks at the Toda chain: explanation of the embedding Toda $\hookrightarrow$ Nahm

(6.13) is the product of two arbitrary *linear* matrix polynomials with *the unit* leading coefficient. This can give us only two arbitrary $p \times p$ matrices. The third one comes from the possibility to multiply spin chain Lax operator by any constant matrix without changing the conservation laws (remind that this is the case, since the r-matrix (4.14) is the permutation operator and, therefore, commutes with the expressions like $U \times U$).

With the established correspondence between $SL(p)$ spin magnet on two sites and the Nahm equations, one can immediately explain the embedding (6.12) of the Toda Lax operator into the Nahm one. Indeed, let us look at the $SU(p)$ pure gauge supersymmetric theory that is described by the brane configuration of Fig.5a corresponding to the SL(2 Toda chain ($SL(2)$ magnet) given on p sites. This configuration can be equally viewed as that depicted in Fig.5b. This latter point of view implies the interpretation of the same system as a specific case of the $SL(p)$ spin magnet given on two sites. This is as stated above equivalent to the Nahm system. Since the $SL(p)$ magnet (= Nahm system) corresponding to the brane configuration of Fig.5b is not of the general form, one should expect that the Toda Lax operator (Fig.5a) is only *embedded* into the Nahm Lax operator. This is, indeed, realized in formula (6.12).

Thus, we have proved that the $SU(2)$ p-charged Nahm system is really equivalent to the 2-site $SL(p)$ spin chain. This hints that the spin chain at many sites is equivalent to the higher group Nahm systems. In the next subsections we check this idea.

6.2 $SU(n)$-monopole, spin chains and generalized Nahm/Diaconescu construction

We start with the description of the $SU(p)$ Nahm construction following [49, 43]. In this case, we fix the large-distance asymptotics of the n components of the Higgs field to be μ_i, $i = 1, ..., n$, $\sum_i \mu_i = 0$. It naturally divides the domain where the Nahm system is given onto $n-1$ intervals (μ_1, mu_2), ..., (μ_{n-1}, μ_n). On each a-th interval there is defined the Nahm system for the three $p_k \times p_k$ matrices $\mathcal{T}_{i,a}(t)$

$$\frac{d\mathcal{T}_{i,a}}{dt} = \frac{1}{2}\epsilon_{ijk}\left[\mathcal{T}_{j,a}, \mathcal{T}_{k,a}\right] \tag{6.14}$$

Thus, $SU(n)$ monopole is defined by the set of charges (topological numbers) $p_1, ..., p_{n-1}$.

The system (6.14) requires some boundary and matching conditions. The details can be found in [49], here we only describe the matching condition at the boundary of two intervals μ_a for the Nahm matrices with some p_a and p_{a-1}. Let us suppose for definiteness that $p_a \geq p_{a-1}$. In the case of equality, all matrices $\mathcal{T}_{i,a}(t)$ and $\mathcal{T}_{i,a-1}(t)$ are analytic nearby μ_a although the condition $\mathcal{T}_{i,a}(t) = \mathcal{T}_{i,a-1}(t)$ around μ_a may be not fulfilled.

In the case of strict inequality, $\mathcal{T}_{i,a-1}(t)$ is analytic as $t \to \mu_a$, while $\mathcal{T}_{i,a}(t)$ has the block form

$$\begin{pmatrix} \mathcal{T}_{i,a-1}(\mu_a) + O(t-\mu_a) & O((t-\mu_a)^{(p_a-p_{a-1}-1)/2}) \\ O((t-\mu_a)^{(p_a-p_{a-1}-1)/2}) & \dfrac{T_{i,a}^{(\mu_a)}}{t-\mu_a} + O(1) \end{pmatrix} \tag{6.15}$$

and $T_{i,a}^{(\mu_a)}$ realizes an irreducible representation of $SU(2)$. These matching conditions have to be added by the boundary conditions $p_0 = p_n = 0$.

Now let us turn to the spectral curve. From the described construction it is clear that, by modulo boundary and matching conditions, the spectral curve is the product of $n-1$ components, each of these being the spectral curve for the $SU(2)$ monopole of charge p_a. The complete curve, therefore, is a curve of the form (6.7) which, in its turn, is a special degenerated curve of the form (4.16) with $p = \sum_a p_a$. In the spin chain language, this means that we are dealing with a very special configuration of the $SL(p)$ magnet given on the n sites.This configuration is determined by "the clasterization" of these n sites into 2-site groups so that there is effectively given the $SL(p_a)$ magnet on each of these groups, with specific matching conditions.

To understand the speciality of this configuration better, we need to turn to the Diaconescu construction [43] that could allow us to propose a natural generalization of the Nahm construction to the generic $SL(p)$ spin chain.

Looking at this construction, one can easily understand what is the specific of the Nahm system that leads to a very peculiar magnet. This specific is obliged to "the locality" of the brane picture in the sense that the 1-branes has their endpoints only on the neighbor 3-branes. This implies that the corresponding spectral curve is just the product of the curves constructed from each pair of the neighbor 3-branes. In terms of spectral curve, this means that there preserved the asymptotic behavior at large w and λ, that is $w \sim \lambda^2$. Thus, appealing to the spin chain interpretation proposed in this section, one can identify the 3-branes with the sites of the spin chain, while the 1-branes gives the $SL(p)$ group, p being the full number of 1-branes. This leads us to the natural conjecture that the generic $SL(p)$ spin chain corresponds to the generalized Diaconescu picture where the 1-branes connects the very left and the very right 3-branes passing through the others (inside) 3-branes. The corresponding Nahm Lax operator is to coincide with the transfer matrix of this spin chain (with Lax operators (4.9) twisted by the constant matrices) and can be represented in the form $L_N(\lambda) = \sum_{i=0}^n \mathcal{T}_i \lambda^i$ with $n+1$ arbitrary matrices $\mathcal{T}_i$.

7 Multiple Λ_i-scales

In this section we introduce a new set of gauge models inspired by some generalization of the periodical Toda chain which contains many Λ type parameters. Let us emphasize from the very beginning that the possibility to be discussed below is purely nonperturbative. The main idea is to consider the correlated inhomogeneous magnetic fluxes and the inhomogeneous "D0 brane lattice" that can be obtained by a nontrivial double scaling limit from the system described by the elliptic spin Calogero model.

Thus, we start with the spin generalization of the elliptic Calogero system that can be the most directly viewed in the Hamiltonian reduction picture [14, 15]. For the simplicity, in this section we discuss the $GL(n)$ system, the reduction to the $SL(n)$ case being trivial and immediate. The system is given by the Lax operator [50]

$$L_{ij}(\lambda) = p_i\delta_{ij} + J_{ij} exp\left((x_i - x_j)\frac{\lambda - \overline{\lambda}}{\tau - \overline{\tau}}\right)\frac{\sigma(\lambda + x_i - x_j)}{\sigma(x_i - x_j)\sigma(\lambda)} \tag{7.1}$$

with the additional requirement $J_{ii} = 0$.[18] Thus, we put $J_{ij} = (1-\delta_{ij})f_{ij}$, $f_{ii} = const$. This latter condition is necessary for integrability of the system (see [51]).

[18] Let us note that generally these diagonal elements are just conserved quantities. The choice made is consistent with the spin Calogero system obtained from the dynamics of poles of the matrix KP system [50].

The additional spin variables in the Lax operator (7.1) f_{ij} satisfy the Poisson brackets

$$\{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj} \tag{7.2}$$

This symplectic form is generally degenerate and its restriction to the symplectic leaves is related to fixing the rank of the matrix f_{ij} to be $l \leq n$ so that the matrix can be presented in the form

$$f_{ij} = \sum_{a=1}^{l} u_i^a v_j^a \tag{7.3}$$

and corresponds to the generic complex orbit $\dfrac{GL(n,\mathbb{C})}{GL(n-l,\mathbb{C}) \times (\mathbb{C}^*)^l}$. The standard spinless Calogero system corresponds to the orbit describing the fundamental representation $\dfrac{GL(n,\mathbb{C})}{GL(n-1,\mathbb{C}) \times (\mathbb{C}^*)}$, i.e. $l = 1$.

The corresponding spectral curve of the spin Calogero system, which generates the conserved quantities, is given, as usual [52], by the determinant form and its genus calculated, say, via Riemann-Roch theorem is $g = np - \frac{p(p+1)}{2} + 1$.

We are not going to discuss here the elliptic system more but instead consider the double scaling procedure that provides us with a set of scales in the low-energy theory after its degeneration to the rational limit. Let us remind how it works in the spinless case. The steps are as follows [53]. First, we degenerate the bare spectral torus $\tau \to i\infty$. Then, to provide the nearest neighbor interaction, we introduce the homogeneous coordinate "lattice" with the large distance Δ between sites

$$x_j = j\Delta + \phi_j \tag{7.4}$$

To see explicitely what kind of interaction emerges in this limit, it suffices to look at the Weirstrass function giving the Calogero potential, although the procedure can be easily repeated for the Lax operator. The Calogero potential has form

$$V(x_{ij}) = g^2 \sum_{i,j} \wp(x_{ij}) \tag{7.5}$$

where $x_{ij} = x_i - x_j$, $g^2 = f_{ij} f_{ji}$ is the (coupling) constant that can be made non-depending on indices by the proper gauge transformation of f_{ij} and the Weierstrass function is defined as

$$\wp(x_{ij}) \equiv \partial^2 \log\sigma(x_{ij}) = \sum_m \frac{1}{\sinh^2(x_{ij} + m\tau)} \tag{7.6}$$

In the limit under consideration we introduce the renormalized coupling constant $g = g_0 exp(\Delta)$ so that g_0 will be ultimately nothing but Λ_{QCD}. Now, choosing $\Delta \sim \tau$ and taking the limit $\tau \to i\infty$, we see that only $m = 0$ term survives in the sum (7.6) in the potential (7.5) so that the resulting potential reads as

$$V_0(\phi_i) = g_0^2 \sum_{j=1}^{n-1} e^{\phi_{i+1} - \phi_i} \tag{7.7}$$

and describes open (non-periodic) Toda chain.

In order to get the periodic Toda chain, one needs to fix $\Delta = \frac{\tau}{n}$. In this case, the $m = -1$ term in (7.6) also contributes into the sum and one finally obtains the potential

$$V_{TC}(\phi_i) = V_0 + g_0^2 e^{\phi_1 - \phi_n} \tag{7.8}$$

describing the periodic Toda chain.

Let us now consider the generic spin Calogero system with $l = n$. In this case, there is a rich spectrum of possibilities so that one would expect n scale parameters instead of a single one. Indeed, it is possible now to introduce the inhomogeneous "lattice" with n independent parameters which give rise to n infrared scales after the independent dimensional transmutation procedure in each $U(1)$ factor. Quantitatively it corresponds to the definition

$$x_{i+1,i} = \Delta_i + \phi_{i+1,i} \tag{7.9}$$

with arbitrary Δ_i which are assumed to be large. The potential in this case has the form

$$V(x_{ij}) = \sum_{i,j} f_{ij} f_{ji} \wp(x_{ij}) \tag{7.10}$$

Consider now different possible limits. In the simplest case we choose

$$f_{i+1,i} = f_{i,i+1} = g_{0,i} \exp(\Delta_i) \tag{7.11}$$

with finite $g_{0,i}$ assuming that no other $f_{i,j}$ have the same exponential factor. This amounts to the open Toda chain with different coupling constants $\Lambda_i \equiv g_{0,i}$ at each site. Again, in order to get the periodicity of the chain, one more constraint has to be imposed

$$\tau = \sum_{i=1}^{n} \Delta_i \tag{7.12}$$

where $f_{n,1} = f_{1,n} \equiv g_{0,n} exp(\Delta_n)$. This leads to the potential

$$V(\phi_{ij}) = \sum_{j=1}^{n-1} g_{0,i}^2 e^{\phi_{i+1}-\phi_i} + g_{0,n}^2 e^{\phi_1-\phi_n} \tag{7.13}$$

However, in this simplest case, we get the same periodic Toda chain, since the difference of the coupling constants in different sites can be removed by shifts of the Toda coordinates.

However, there are less trivial limits given by the choice

$$f_{i+j,i} = g_{0,j} \exp(\Delta_j) \tag{7.14}$$

that is to some extent complimentary to (7.11). This choice, with specially adjusted Δ_i's and the proper periodicity condition, leads to a "more filled" Lax operator than Toda one, with several non-zero diagonals so that more pairs ϕ_i, ϕ_k interact. In fact, this Lax operator is similar to (5.16) and, in the most general situation, is "completely filled". This generic system, certainly, gives rise to the system that differ from the Toda one but the number of free parameters still can not exceed n ($n-1$ in the $SL(n)$ case).

Turn now to the brane picture behind the described limits. Let us begin with the rough "perturbative" configuration. Remind that for the single scale $N = 2$ theory one has a pair of NS5 brane with n D4 branes between them. The distance in the x^6 direction is identified with $\frac{1}{g^2}$. Therefore, in order to reproduce the above consideration, one would order the D0-branes along the x^6-coordinate in the simple Toda case but in the general case the D0-branes are to be assigned with arbitrary positions along this direction. This provides a set of new scales in field theory. The corresponding multiscale low-energy theory is broken, apart from the usual Higgs mechanism of the gauge $SU(n)$ symmetry breaking, by the explicit gauge symmetry breaking at the regulator scale. Due to the asymptotical freedom of the theory and the conformal anomaly, this ultraviolet breaking is lifted down to the low energy sector. In the Lagrangian terms this means that we break $N = 4$ theory down to $N = 2$ adding more complicated regulators with the set of mass parameters that manifestly break the gauge symmetry Tr $\left[M(\Phi_1^2 + \Phi_2^2)\right]$, where M is a mass regulator matrix.

Another interpretation of the regulator scales is related to the recent proposal [54] that the UV cut-offs can be identified with the momenta p_{11} in M theory. Therefore, we can describe our multiscale case as corresponding to the nondegenerated set of momenta along the eleventh dimension. This implies the possibility of nontrivial brane scattering processes with the p_{11} transfer [55] which should has much to do with the interpretation of Whitham dynamics as a scattering process [44].

Certainly, entering the several UV scales into the low-energy physics deserves further studies and we are going to return to this question in the forthcoming paper [56].

8 Discussion on symmetries

This section is slightly outside the main line of the paper, but we consider it important for the future development. In fact, we speculate on what might be the role of some key notions/properties of integrable systems

that have not been used yet in the SUSY theories/brane framework. In particular, we briefly discuss the hidden symmetries which are well known as "non-abelian symmetries" in the context of the integrable field theories and have some counterpart in the finite dimensional case. Their role in the classical dynamics on the spectral curve, is still to be understood but they certainly are of the great importance at the quantum level[19]. However, we start with some discussion of how the finite-dimensional system is embedded into $2d$ one and only then come to the symmetry issues.

8.1 Embedding in 2d integrable systems

In order to see the full (typically infinite dimensional) symmetry group underlying a finite-dimensional system, one needs to embed it into a ($2d$) infinite dimensional integrable theory. Embedding of such a type is known, in particular, for the systems of the Calogero type. In fact this is a heuristic observation that there is a close relation of these systems with special solutions in 2d integrable field theories [60]. This correspondence has no rational explanation yet.

We expect that the embedding of the many-body systems related to the SUSY/brane theories into $2d$ systems would be very useful in understanding of the processes of creation/annihilation etc. To be precise this embedding is formulated as follows [61]. One should start with the rational, trigonometric or elliptic solutions to the KP hierarchy with the τ-function that has a multiplicative form

$$\tau(x,t) = \prod_{i=1,\ldots,n} \sigma(x - x_i(t)) \tag{8.1}$$

Now the condition for this τ-function to satisfy the equation of the KP hierarchy defines the dependence of zeros x_i on all the times of the hierarchy. This dependence is determined by the dynamics given by the Hamiltonians of the many-body Calogero system, that is to say, by the traces of the Calogero Lax operator corresponding to the, accordingly, rational, cylinder or elliptic geometry. The number of factors in the τ-function coincides with the number of particles in the Calogero system. Two more requirements are to be added – the Calogero coupling constant is to be fixed and is not allowed to take arbitrary values and the evolution has to be started with the specific "locus" particle configuration with zero particle momenta. This correspondence holds for the generalization of the KP system to the $2d$ Toda lattice hierarchy, the dynamics of the specific solutions to the $2d$ Toda being governed by Hamiltonians of the Ruijsenaars finite-dimensional integrable system [62].

Let us remark that according to [60] one can establish an analogous correspondence between the Calogero and KdV dynamics. In this case, the whole procedure can be naturally performed in two steps. First, one fix the system of n coinciding particles with the total charge n, which further decays (in accordance with the KdV dynamics) into the zero momenta "locus" configuration that evolves with the Calogero Hamiltonians. This locus configuration is not arbitrary but possesses $\mathbb{Z}_n$-symmetry. This procedure can be also interpreted in brane terms, with the Calogero degrees of freedom regarded as D0 coordinates. Then, the initial configuration is interpreted as a bound state of n D0-branes which KdV-evolve to the stable "locus" configuration which then evolves with the Calogero Hamiltonians.

Within the described picture, the possible role that $2d$ integrable systems like KP or Toda could play looks very promising. Indeed, we have seen that the dynamics of the finite number of D0-branes corresponds to the evolution of the very special solution of $2d$ theory. Therefore, it seems unavoidable that the general $2d$ dynamics is described by the nontrivial D0 distribution along the x^6 direction. Thus, the integrable system to some extend looks as a second quantized "brane or string" theory. Note that the KdV dynamical variable u(x) now can be treated as the x^6-component of the stress energy tensor so that the expression for the simplest (rational) solution

$$u(x) = \sum_i \frac{1}{(x - x_i(t))^2}$$

[19] Let us fix at the very beginning that all the hidden symmetries under consideration are the Poisson-Lie ones.

in D0 brane terms looks quite satisfactory.

This second quantized picture can be pushed even further. To this end, let us note that embedding different Calogero systems into the same KdV (or, more generally, KP/$2d$ Toda lattice) hierarchy plays a unifying role. Indeed, solutions of the Calogero systems with different number of particles, that is to say, with different gauge group ranks, being considered as solutions to the same KdV hierarchy are related by a Bäcklund transformation that can be realized by the (fermionic) vertex operators [63]. Thus, these vertex operators are associated with the operator of the brane creation providing the change of the gauge group rank. This means that the (sub)space of the solutions to the larger hierarchy can be chosen as a configuration space of all the vacua/string theories. In its particular realization, this space can be described as an infinite dimensional Grassmannian [64] and reminds of the analogous construction known from the perturbative string physics [65].

In the rest part of this section we discuss the issues of symmetries in integrable systems that can be easier and more naturally formulated for the quantum systems. Although so far we have met only the classical integrable systems, below we follow the general approach equally applicable to both quantum and classical integrable systems. Among other advantages, this will make quantization of the construction an immediate thing.

Let us give some general comments concerning the quantum picture. From the consideration above it is clear that the integrable dynamics would proceed on the instantonic moduli space so all the ingredients of the quantum picture should acquire the meaning of the appropriate objects on this moduli space. Note that in the course of quantization, i.e. constructing the wave function, in the present "Hitchin treatment" we need to neglect the Whitham variables taking them as just parameters. This corresponds to the Born-Oppenheimer approximation. It is just in this approximation the whole wave function reflects the structure of the instanton moduli space.

As we have argued, instead of the infinite dimensional instanton moduli space, we work with the finite number of effective variables which substitute the infinite instanton sums. The wave functions seem to be related to the cohomology of the moduli space while the quantum partition function would play the role of the generation function for the intersection numbers in cohomologies.

Asymptotics of the wave function gives the S-matrix for the effective degrees of freedom – D0 branes. The integrability and the presence of the additional conservation laws promise the exactness of the S-matrix as well as its pure pole structure. At the same time, in order to derive the form-factors (in our context, it corresponds to the form-factor of the compound state of multiple D0 branes), one needs to use the hidden non-abelian symmetries which we are going to discuss now.

8.2 Bäcklund transformation

The remnant of the infinite dimensional non-abelian symmetry in the Toda system is the Bäcklund transformation. It is known that the nontrivial Bäcklund transformation for the periodic Toda chain corresponds to the transition between two equivalent sets of variables x_{2n} and x_{2n+1}. At the quantum level, it is convenient to consider the wave function in the action-angle representation which is the solution to the Baxter equation

$$Q(\lambda+i)+\Lambda_{QCD}^{2n_c}Q(\lambda-i)=Q(\lambda)TrT(\lambda) \tag{8.2}$$

where $T(\lambda)$ is the transfer matrix of the Toda chain and λ is the spectral parameter. This equation actually is the quantum counterpart of the spectral curve equation considered as the operator acting on the wave function in the separated variables [57][20].

The key point is that the function Q turns out to be the generating function for the canonical Bäcklund transformation. Namely, it can be related [58] to the kernel of the operator acting between two set of variables

[20]In the classical limit, the functions $\frac{Q(\lambda+i)}{Q(\lambda)}$ and $\frac{Q(\lambda-i)}{Q(\lambda)}$ can be identified with w and $\frac{1}{w}$ respectively.

$q_i, \overline{q_i}$. Explicit form of the integral kernel of $Q(\lambda)$ looks as follows

$$Q_\lambda(q|\overline{q}) = \prod_j W_\lambda(\overline{q_j} - q_j)\overline{W_\lambda}(q_j - \overline{q_{j+1}}) \tag{8.3}$$

where

$$\begin{aligned} W_\lambda(q) &= exp(i\lambda q - e^q) \\ \overline{W_\lambda}(q) &= exp(-i\lambda q - e^q) \end{aligned} \tag{8.4}$$

Logarithms of the matrix elements of Q are the generating functions for the canonical Bäcklund transformation. This is in agreement with the general viewpoint: Q is actually related to the Toda τ-function while the τ-function is the generating function for some canonical transformations.

8.3 Semiclassical Yangian type symmetry

Let us turn now to the analogous symmetry in the integrable system corresponding to the $N = 2$ SQCD – spin chain [59]. It is known that the spin chains enjoy two different sets of the conservation laws – Abelian and non-Abelian ones (they are local and non-local correspondingly). The charges of the both sets are conserved but, because of the non-abelian nature of the second set, one can not diagonalize the Yangian charges simultaneously. The explicit formulae for the Yangian structure can be found in [59]. Hereafter, we consider the $SL(2)$ spin chain, it is homogeneous and the Casimirs at all sites are the same. This is equivalent to the coincidence of all masses of the fundamental matter in the $N = 2$ SQCD.

We now emphasize that the knowledge of charges is equivalent to the data of the monodromy matrix T (we use here the unit normalization of the determinant):

$$T(\lambda) = \begin{pmatrix} A(\lambda)B(\lambda) \\ C(\lambda)D(\lambda) \end{pmatrix} \qquad ; \qquad \det T(\lambda) = AD - BC = 1 \tag{8.5}$$

Yangian charges can be expressed in terms of the monodromy matrix as follows

$$\begin{aligned} \mathcal{P}_{ij}(\lambda) &= \delta_{ij} + \sum_{n=0}^{\infty} \lambda^{-n-1}\mathcal{P}_{ij}^n = \frac{1}{2} tr(T\sigma_i T^{-1}\sigma_j) \\ \mathcal{P}_i(\lambda) &= \sum_{n=0}^{\infty} \lambda^{-n-1}\mathcal{P}_i^n = \epsilon_{ijk}\mathcal{P}_{jk}(\lambda) \quad\quad i = 1,2,3 \end{aligned} \tag{8.6}$$

Therefore, the generators $\mathcal{P}_{ij}(\lambda)$ and $\mathcal{P}_i(\lambda)$ are quadratic functions of the matrix elements of $T(\lambda)$.

It is also possible to express $T(\lambda)$ in terms of the generating functions $\mathcal{P}_i(\lambda)$:

$$T(\lambda) = \frac{1}{2}W(\lambda)\mathbf{1} - \frac{i}{2}W^{-1}(\lambda)\sum_i \mathcal{P}_i(\lambda)\sigma_i \tag{8.7}$$

with $W(\lambda) = \sqrt{2 + \sqrt{4 - \vec{\mathcal{P}}^2(\lambda)}}$.

Thus, the charges $\mathcal{P}_i(\lambda)$ contain the same amount of information as the monodromy matrix. There are also the following relations between the $\mathcal{P}_i(\lambda)$ and the matrix elements of $T(\lambda)$:

$$\begin{aligned} \mathcal{P}_+(\lambda) &= \mathcal{P}_1(\lambda) + i\mathcal{P}_2(\lambda) = 2iW(\lambda)\ C(\lambda) \\ \mathcal{P}_-(\lambda) &= \mathcal{P}_1(\lambda) - i\mathcal{P}_2(\lambda) = 2iW(\lambda)\ B(\lambda) \\ \mathrm{Tr}T(\lambda) &= W(\lambda) = W\left[\vec{\mathcal{P}}^2(\lambda)\right] \end{aligned} \tag{8.8}$$

Therefore, $\vec{\mathcal{P}}^2(\lambda)$ is also a generating function for (non-local) commuting quantities.

For the completeness, let us manifestly write down the Poisson brackets of semiclassical Yangian generators

$$\begin{aligned}\{\mathcal{P}_i^0, \mathcal{P}_j^0\} &= 4\epsilon_{ijk}\mathcal{P}_k^0 \\ \{\mathcal{P}_i^0, \mathcal{P}_j^1\} &= 4\epsilon_{ijk}\mathcal{P}_k^1 \\ \{\mathcal{P}_i^1, \{\mathcal{P}_j^1, \mathcal{P}_k^0\}\} - \{\mathcal{P}_i^0, \{\mathcal{P}_j^1, \mathcal{P}_k^1\}\} &= A_{ijk}^{lmn}\mathcal{P}_l^0\mathcal{P}_m^0\mathcal{P}_n^0 \\ \{\{\mathcal{P}_i^1, \mathcal{P}_j^1\}, \{\mathcal{P}_k^0, \mathcal{P}_l^1\}\} + \{\{\mathcal{P}_k^1, \mathcal{P}_l^1\}, \{\mathcal{P}_i^0, \mathcal{P}_j^1\}\} &= 8(A_{ija}^{mnp}\epsilon_{kla} + A_{kla}^{mnp}\epsilon_{ija})\mathcal{P}_m^0\mathcal{P}_n^0\mathcal{P}_p^1\end{aligned} \tag{8.9}$$

with $A_{ijk}^{lmn} = \frac{2}{3}\epsilon_{ila}\epsilon_{jmb}\epsilon_{knc}\epsilon^{abc}$.

One sees that these relations form a deformation of those defining the Borel part of the $su(2)$ loop algebra. All the non-local charges $\mathcal{P}_{ij}^n$ can be expressed in terms of the Poisson brackets between the two first charges $\mathcal{P}_i^0$ and $\mathcal{P}_i^1$. Therefore, the whole Poisson algebra of the symmetries is generated by these two charges. They are nothing but the corresponding Chevalley generators.

Finally, let us describe how the monodromy matrix generates the transformations of the spin variable on the k-th site which corresponds in the brane language to the action on the k-th (D6+D4) brane. By an explicit computation of the Poisson brackets between the monodromy matrix and the spin variables, one can show that the variation $\delta_v^n S(k)$ of the spin variables reads as:

$$\delta_i^n S(k) \equiv \{\mathcal{P}_i^n, S(k)\} = \; i \oint \frac{d\lambda}{2i\pi}\lambda^n \;\mathrm{Tr}_1\Big(\;(\sigma_i T^{-1}(\lambda)\otimes 1)\Big\{T(\lambda)\otimes 1, 1\otimes S(k)\Big\}\;\Big) \tag{8.10}$$

Here Tr_1 denotes the trace over the first space in the tensor product and v_i is the parameter of the transformation. This formula describes the transformations $S(k) \to \delta_v^n S(k)$ and indicates that $T(\lambda)$ also generates the non-Abelian symmetry.

Let us comment now the possible meaning of the non-abelian symmetry within the brane approach. Usually this symmetry is important at the quantum level providing the set of restrictions onto the S-matrix. In our context it would mean that the symmetry restricts the brane scattering. Note that the Yangian charges are essentially non-local. In the SQCD context this would correspond to mixing the different flavors (that are associated with different sites of chain) and presumably might be relevant for the barionic branch of the theory.

9 Conclusion

In the present paper we discussed effective $N = 2$ SUSY field theories, corresponding integrable systems, monopoles and their stringy analogues. The equivalences were established at low energies in field theories as well as in string theory models. One can try to go further and ask whether such connections persist if we consider higher derivatives in field theory corresponding to non-lowest excitations in string theory. The answer seems to be yes, and there is natural candidates in the world of integrable systems for the counterpart of higher-derivative terms. They are related to the higher times and introducing two-dimensional integrable systems. We are going to elaborate this point in the subsequent publication [56]. We also postponed the discussion of elliptic models (and their degenerations) as well as five- and six-dimensional field theories (XXZ and XYZ spin chains), which constitute the two last rows in Fig.1. The new examples confirm the validity of the general approach and suggest the proper way to include additional degrees of freedom. Moreover, it provides the possibility to get completely new viewpoints in the field theory setup which we have seen in section 7. Nevertheless, despite many supporting arguments including the identification of branes as the proper degrees of freedom, the clear *physical* answer to the question: "Why the low-energy SYM is governed by integrable system?" is still missed.

The answer to this question requires the knowledge of the exact derivation of the effective degrees of freedom in the integrable systems from the instantonic sums. We have argued that somehow the instantons are summed into the finite number of degrees of freedom. However, the proper derivation and explanation of

this phenomena from the path integral of SYM is still lacking. The answer would imply a nontrivial localization while integrating over the instanton moduli space.

So far only the tiny part of the "integrability world" has been recognized in the supersymmetric YM theories. Say, the issues of underlying symmetries especially in connection with the quantization problem as well as interpretation of the non-Abelian "spectrum generating" symmetries in the brane language have been out of mainstream. We are planning to discuss this points elsewhere.

Another important point about the integrable treatment is to interpret the wave functions (which would be interpreted as the correlators in a topological theory) that should provide the important information about the vacuum configurations. The wave functions actually serve as generating functions for the canonical transformations. It is interesting to identify these canonical transformations in the brane language.

Let us say some words about the possible role of integrability in $N = 1$ SUSY theory. It was recently shown [39, 66] that the perturbative spectral curve in such a theory is a sphere with marked points which suggests that we leave the family of the "finite-gap" solutions. But it can be not the whole story. Indeed, the branching points on the spectral curves come from the Coulomb branch moduli and so Λ_{QCD} does. The Coulomb moduli degenerate to the set of points which means that from the $N = 2$ SUSY point of view we can no longer vary the integrals of motion. But the potential cut in the strong coupling region still persists due to Λ_{QCD} so the one-gap solutions are potentially possible.

Recently another object from the integrability cuisine was also discussed in the context of $N = 1$ theory, that is, the "creek equation" from [67] that gives the profile of the BPS objects. This equation is a direct counterpart of the Nahm equation treatment above. So the would be integrable system in $N = 1$ context corresponds to the dynamics of several interacting BPS states .

We have seen some similarities with the "confining string" scenario above. Indeed, one of the main points of that scenario is the presence of "the monopole ring" in the vacuum state, the appearance of the effective "confining string" to substitute the infinite number of instanton and the appearance of some auxiliary surface whose genus corresponds to the rank of the gauge group. One immediately recognizes these ingredients in the integrability approach but we do not know any real statements behind this observation.

Note that the picture considered in the paper has, in fact, some solid state analogies. There are many examples in the solid state physics when the mass gap in the spectrum of the quasiparticles vanishes on (hyper)surfaces of different dimensions in momentum space. Just these "defects" ("monopoles") in the momentum space are counterparts of branes in our approach. The very appearance of the (topologically nontrivial) hypersurfaces in the theories with anomalies has a simple interpretation. Indeed, the presence of the anomaly (for instance, the conformal anomaly in the SYM theory) implies the nontrivial level crossing that can be seen within the standard Berry phase treatment. In our consideration the mass of regulator (entering the low-energy sector due to the conformal anomaly) gives rise to the "monopole field" in the space of fields in agreement with this interpretation.

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10 Appendix. $SL(3)$ chain

In this Appendix we consider the simple case of the $SL(3)$ chain to give explicit examples of some formulas of sect.3.

$SL(3)$ spin chain. The Lax operator is given by the formula

$$L_i = \begin{pmatrix} S^{(1)}_{0,i} + \lambda + \lambda_i & S^{(1)}_{-,i} & S^{(12)}_{-,i} \\ S^{(1)}_{+,i} & S^{(2)}_{0,i} + \lambda + \lambda_i & S^{(2)}_{-,i} \\ S^{(12)}_{+,i} & S^{(2)}_{+,i} & -S^{(1)}_{0,i} - S^{(2)}_{0,i} + \lambda + \lambda_i \end{pmatrix} \tag{A.1}$$

The spectral curve in this case has the form

$$w^3 + \mathrm{Tr}T(\lambda)w^2 + \sum_i \det M_i(\lambda) w + \det T(\lambda) = 0 \tag{A.2}$$

where $T(\lambda)$ is, as before, the transfer matrix and $M_i(\lambda)$ is matrix obtained from the transfer matrix by removing the i-th column and i-th row. In this reference integrable system, $J_1 = 1$ and $J_2 = \det T(\lambda) = \prod_i \det L_i(\lambda)$. To determine mass spectrum we need to calculate $\det L_i(\lambda)$. It has the form

$$\det L_i(\lambda) = (\lambda + \lambda_i)^3 - C_2^{(i)}(\lambda + \lambda_i) - C_3^{(i)} \tag{A.3}$$

where Casimir functions are

$$\begin{gathered} C_2^{(i)} = S^{(1)}_{+,i}S^{(1)}_{-,i} + S^{(2)}_{+,i}S^{(2)}_{-,i} + S^{(12)}_{+,i}S^{(12)}_{-,i} + \left(S^{(1)}_{0,i}\right)^2 + \left(S^{(2)}_{0,i}\right)^2 + S^{(1)}_{0,i}S^{(2)}_{0,i} \\ C_3^{(i)} = S^{(12)}_{-,i}S^{(12)}_{+,i}S^{(2)}_{0,i} + S^{(2)}_{-,i}S^{(2)}_{+,i}S^{(1)}_{0,i} - S^{(1)}_{+,i}S^{(2)}_{+,i}S^{(12)}_{-,i} - S^{(1)}_{-,i}S^{(2)}_{-,i}S^{(12)}_{+,i} + \\ + S^{(1)}_{0,i}S^{(2)}_{0,i}\left(S^{(1)}_{0,i} + S^{(2)}_{0,i}\right) - S^{(1)}_{-,i}S^{(1)}_{+,i}\left(S^{(1)}_{0,i} + S^{(2)}_{0,i}\right) \end{gathered} \tag{A.4}$$

that can be manifestly checked using the Poisson brackets

$$\begin{gathered} \{S^{(1)}_+, S^{(1)}_-\} = S^{(2)}_0 - S^{(1)}_0, \quad \{S^{(2)}_+, S^{(2)}_-\} = -S^{(1)}_0 - 2S^{(2)}_0, \quad \{S^{(1)}_\pm, S^{(1)}_0\} = \pm S^{(1)}_\pm, \\ \{S^{(1)}_\pm, S^{(2)}_0\} = \mp S^{(1)}_\pm, \quad \{S^{(2)}_\pm, S^{(1)}_0\} = 0, \quad \{S^{(2)}_\pm, S^{(2)}_0\} = S^{(2)}_\pm, \end{gathered} \tag{A.5}$$

and defining relations

$$\{S^{(1)}_\pm, S^{(2)}_\pm\} = \mp S^{(12)}_\pm \tag{A.6}$$

so that

$$\begin{gathered} \{S^{(1)}_-, S^{(12)}_+\} = -S^{(2)}_+, \quad \{S^{(2)}_-, S^{(12)}_+\} = S^{(1)}_+, \quad \{S^{(1)}_+, S^{(12)}_-\} = S^{(2)}_-, \quad \{S^{(2)}_+, S^{(12)}_-\} = -S^{(1)}_-, \\ \{S^{(12)}_\pm, S^{(1)}_0\} = \pm S^{(12)}_\pm, \quad \{S^{(12)}_\pm, S^{(2)}_0\} = 0, \quad \{S^{(12)}_-, S^{(12)}_+\} = 2S^{(1)}_0 + S^{(2)}_0 \end{gathered} \tag{A.7}$$

Generally, the determinant (A.3) has three distinct roots (masses), since we deal with the reference system, that is to say, $J_1(\lambda) = 1$. In order to obtain non-unit $J_1(\lambda)$, one needs to consider the case of, two coinciding roots. This condition implies specialization of the general orbit

$$4C_2^3 = 27C_3^2 \tag{A.8}$$

and the coinciding masses are equal to

$$m_c = -\lambda_i \pm \sqrt{\frac{C_2}{3}} \tag{A.9}$$

Then, the factor $(\lambda - m_c)^2$ is nothing but $J_1(\lambda)$ ($J_1(\lambda)$ of the general form can be obtained by degenerating at a set of sites). Now one can check that condition (A.8) leads to factoring out the multiplier $(\lambda - m_c)$ in the coefficient in front of w in (A.2) to reproduce the correct dependence of the spectral curve on $J_1(\lambda)$. This coefficient is equal to

$$T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33} - T_{12}T_{21} - T_{13}T_{31} - T_{23}T_{32} \tag{A.10}$$

For the definiteness, let us consider the degeneration at the first site. Note that (A.10) can be rewritten as

$$\begin{aligned}&(L_{1,11}L_{1,22}-L_{1,12}L_{1,21})\left(\widehat{T}_{11}\widehat{T}_{22}-\widehat{T}_{12}\widehat{T}_{21}\right)+\\&+(L_{1,11}L_{1,33}-L_{1,13}L_{1,31})\left(\widehat{T}_{11}\widehat{T}_{33}-\widehat{T}_{13}\widehat{T}_{31}\right)+\\&+(L_{1,22}L_{1,33}-L_{1,23}L_{1,32})\left(\widehat{T}_{22}\widehat{T}_{33}-\widehat{T}_{23}\widehat{T}_{32}\right)\end{aligned} \tag{A.11}$$

where $\widehat{T}$ is the transfer matrix without the first site and $L_{1,ij}$ is (i,j)-matrix element of L_1. This expression can be rewritten as follows

$$\begin{aligned}&\left(\lambda+\lambda_i+\frac{1}{2}\left(S_{0,1}^{(1)}+S_{0,1}^{(2)}\right)+\frac{1}{2}\sqrt{\left(S_{0,1}^{(1)}-S_{0,1}^{(2)}\right)^2+4S_{-,1}^{(1)}S_{+,1}^{(1)}}\right)\times\\&\times\left(\lambda+\lambda_i+\frac{1}{2}\left(S_{0,1}^{(1)}+S_{0,1}^{(2)}\right)-\frac{1}{2}\sqrt{\left(S_{0,1}^{(1)}-S_{0,1}^{(2)}\right)^2+4S_{-,1}^{(1)}S_{+,1}^{(1)}}\right)\times\\&\times\left(\widehat{T}_{11}\widehat{T}_{22}-\widehat{T}_{12}\widehat{T}_{21}\right)+\\&+\left(\lambda+\lambda_i-\frac{1}{2}S_{0,1}^{(2)}+\frac{1}{2}\sqrt{\left(S_{0,1}^{(2)}+2S_{0,1}^{(1)}\right)^2+4S_{-,1}^{(12)}S_{+,1}^{(12)}}\right)\times\\&\times\left(\lambda+\lambda_i-\frac{1}{2}S_{0,1}^{(2)}-\frac{1}{2}\sqrt{\left(S_{0,1}^{(2)}+2S_{0,1}^{(1)}\right)^2+4S_{-,1}^{(12)}S_{+,1}^{(12)}}\right)\times\\&\times\left(\widehat{T}_{11}\widehat{T}_{33}-\widehat{T}_{13}\widehat{T}_{31}\right)+\\&+\left(\lambda+\lambda_i+\frac{1}{2}S_{0,1}^{(1)}+\frac{1}{2}\sqrt{\left(S_{0,1}^{(1)}+2S_{0,1}^{(2)}\right)^2+4S_{-,1}^{(2)}S_{+,1}^{(2)}}\right)\times\\&\times\left(\lambda+\lambda_i+\frac{1}{2}S_{0,1}^{(1)}-\frac{1}{2}\sqrt{\left(S_{0,1}^{(1)}+2S_{0,1}^{(2)}\right)^2+4S_{-,1}^{(2)}S_{+,1}^{(2)}}\right)\times\\&\times\left(\widehat{T}_{22}\widehat{T}_{33}-\widehat{T}_{23}\widehat{T}_{32}\right)\end{aligned} \tag{A.12}$$

Now by the straightforward calculation one can check that each of three terms in this formula is proportional to $(\lambda-m_c)$ provided the condition (A.8) is satisfied.

$SL(3)$-Toda-like system. Now let us consider the "limiting" degenerations of the Lax operator (A.1) analogous to the Toda system in the $SL(2)$ case. We follow $SL(2)$ case and multiply the Lax operator (A.1) by the constant diagonal matrix U. In this case, there are two possibilities. First possibility is to choose

$$U=\begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&1\end{pmatrix} \tag{A.13}$$

and redefine $S_+^{(1)}\to\frac{1}{\alpha}S_+^{(1)}$, $S_-^{(2)}\to\frac{1}{\alpha}S_-^{(2)}$. Now, bringing α to zero, one obtains the Lax operator (hereafter, we omit inhomogeneity, since it does not effect the result)

$$L=\begin{pmatrix}S_0^{(1)}+\lambda&S_-^{(1)}&S_-^{(12)}\\S_+^{(1)}&0&S_-^{(2)}\\S_+^{(12)}&S_+^{(2)}&-S_0^{(1)}-S_0^{(2)}+\lambda\end{pmatrix} \tag{A.14}$$

Then, instead of (A.5), we get

$$\begin{gathered}\{S_+^{(1)},S_-^{(1)}\}=\{S_+^{(2)},S_-^{(2)}\}=0,\quad \{S_\pm^{(1)},S_0^{(1)}\}=\pm S_\pm^{(1)},\quad \{S_\pm^{(1)},S_0^{(2)}\}=\mp S_\pm^{(1)},\\ \{S_\pm^{(2)},S_0^{(1)}\}=0,\quad \{S_\pm^{(2)},S_0^{(2)}\}=\pm S_\pm^{(2)}\end{gathered}\tag{A.15}$$

and instead of (A.6) –

$$\{S_\pm^{(1)},S_\pm^{(2)}\}=0\tag{A.16}$$

while (A.7) remain unchanged (now, however, one can not use the defining relations (A.6) to generate all other (A.7) but they should be given independently).

At the next step, the second Casimir function of the algebra (A.15)-(A.16) $S_+^{(1)}S_-^{(1)}+S_+^{(2)}S_-^{(2)}$ should be put equal to zero. Then the determinant of the Lax operator (A.14) is equal to the third Casimir function

$$S_+^{(1)}S_+^{(2)}S_-^{(12)}+S_-^{(1)}S_-^{(2)}S_+^{(12)}+S_-^{(1)}S_+^{(1)}\left(2S_0^{(1)}+S_0^{(2)}\right)\tag{A.17}$$

that can be chosen equal to unity.

The second possibility of degenerations of the Lax operator (A.1) looks slightly simpler having less non-trivial entries. It can be achieved by multiplying the Lax operator by the matrix U of the form

$$U=\begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&\alpha\end{pmatrix}\tag{A.18}$$

This form of the matrix allows one to redefine, apart from $S_+^{(1)}\to\frac{1}{\alpha}S_+^{(1)}$, $S_\pm^{(2)}\to\frac{1}{\alpha}S_\pm^{(2)}$ $S_+^{(12)}\to\frac{1}{\alpha}S_+^{(12)}$, also the diagonal element $S_0^{(2)}\to\frac{1}{\alpha}S_0^{(2)}$ *without* tuning the inhomogeneity (compare with the $SL(2)$ case). Now taking α to zero, one obtains the Lax operator

$$L=\begin{pmatrix}\lambda+S_0^{(1)}&S_-^{(1)}&S_-^{(12)}\\S_+^{(1)}&S_0^{(2)}&S_-^{(2)}\\S_+^{(12)}&S_+^{(2)}&-S_0^{(2)}\end{pmatrix}\tag{A.19}$$

and algebra of the Poisson brackets

$$\begin{gathered}\{S_+^{(1)},S_-^{(1)}\}=S_0^{(2)},\quad \{S_+^{(2)},S_-^{(2)}\}=0,\quad \{S_\pm^{(1)},S_0^{(1)}\}=\pm S_\pm^{(1)},\quad \{S_\pm^{(1)},S_0^{(2)}\}=\{S_\pm^{(2)},S_0^{(1)}\}=0,\\ \{S_\pm^{(2)},S_0^{(2)}\}=0,\quad \{S_\pm^{(1)},S_\pm^{(2)}\}=0,\quad \{S_-^{(1)},S_+^{(12)}\}=-S_+^{(2)},\quad \{S_-^{(2)},S_+^{(12)}\}=\{S_+^{(2)},S_-^{(12)}\}=0,\\ \{S_+^{(1)},S_-^{(12)}\}=S_-^{(2)},\quad \{S_\pm^{(12)},S_0^{(1)}\}=\pm S_\pm^{(12)},\quad \{S_\pm^{(12)},S_0^{(2)}\}=0,\quad \{S_-^{(12)},S_+^{(12)}\}=2S_0^{(1)}+S_0^{(2)}\end{gathered}\tag{A.20}$$

One can easily check that $S_0^{(2)}$ and $S_\pm^{(2)}$ are the Casimir functions of the algebra (A.20). In order to get constant determinant of the Lax operator (A.19) we put $S_\pm^{(2)}=\pm1$ and $S_0^{(2)}=1$.Finally this leads us to the Lax operator

$$L=\begin{pmatrix}\lambda+S_0^{(1)}&S_-^{(1)}&S_-^{(12)}\\S_+^{(1)}&1&-1\\S_+^{(12)}&1&-1\end{pmatrix}\tag{A.21}$$

and the Poisson bracket algebra

$$\{S_+^{(1)},S_-^{(1)}\}=1,\quad \{S_\pm^{(1)},S_\mp^{(12)}\}=-1,\quad \{S_\pm^{(1)},S_0^{(1)}\}=\pm S_\pm^{(1)},\quad \{S_\pm^{(12)},S_0^{(1)}\}=\pm S_\pm^{(12)}\tag{A.22}$$

The determinant of the Lax operator (A.21) is equal to the Casimir function

$$C=S_+^{(1)}S_-^{(1)}+S_+^{(1)}S_-^{(12)}-S_+^{(12)}S_-^{(12)}-S_-^{(1)}S_+^{(12)}\tag{A.23}$$

and also can be put equal to unity.

Let us note that (A.21) describes a slightly specified case of the general situation, since the constant sub-matrix $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ in the Lax operator is chosen to be traceless. It can be avoided by a proper rescaling of $S_0^{(1)}$ and inhomogeneity. This particular choice is one that admits the $n \times n$ representation. As any $n \times n$ representation for the $SL(p)$, $p > 2$ magnet, it describes a degenerated spectral curve as compared to the general pure gauge theory (see s.5.3). Let us trace out how it happens in this concrete case.

First, one can observe that the system of the Poisson brackets (A.22) has Casimir functions $S_\pm^{(1)} \mp S_\pm^{(12)}$. Taking into account that we put Casimir function (A.23) to be unit, we can choose $S_\pm^{(12)} = \pm S_\pm^{(12)} \mp c_\pm$, $c_+c_- = 1$. The simplest choice is $c_+ = c_- = 1$. Then, literally following s.5.1, one gets the $n \times n$ representation:

$$\mathcal{L}(w) = \begin{pmatrix} S_{0,1}^{(1)} & 1 & 0 & & \frac{1}{w} & \frac{S_{+,n}^{(1)} + S_{-,n}^{(1)} - 1}{w} \\ S_{+,2}^{(1)} + S_{-,2}^{(1)} - 1 & S_{0,2}^{(1)} & 1 & \dots & 0 & \frac{1}{w} \\ 1 & S_{+,3}^{(1)} + S_{-,3}^{(1)} - 1 & S_{0,3}^{(1)} & \dots & & 0 \\ & & & \ddots & & \\ & & \dots & & \ddots & \vdots \\ -w & 0 & 0 & & & S_{0,n}^{(1)} \end{pmatrix} \tag{A.24}$$

with the following Poisson brackets: $\{S_{+,i}^{(1)}, S_{-,j}^{(1)}\} = \delta_{ij}$, $\{S_{\pm,i}^{(1)}, S_{0,j}^{(1)}\} = \pm S_{\pm,i}^{(1)}\delta_{ij}$ that can be realized by the two harmonic oscillators $\{p_a, q_b\} = \delta_{ab}$, $a, b = 1, 2$: $S_+^{(1)} = p_1 e^{q_2}$, $S_-^{(1)} = q_1 e^{-q_2}$, $S_0^{(1)} = p_2$.

The Lax operator (A.24) leads to the following spectral curve (if $n > 2$)

$$w + P_n^{(1)}(\lambda) + P_{n-2}^{(2)}(\lambda)\frac{1}{w} + \frac{1}{w^2} = 0 \tag{A.25}$$

In fact, the low diagonal of units in the Lax operator is nothing but the product of c_+c_-. Requiring this product to be zero, we can reproduce the particular Lax operator of the construction [38] (see s.5.2) and the spectral curve takes the form

$$w + P_n^{(1)}(\lambda) + P_{n-2}^{(2)}(\lambda)\frac{1}{w} = 0 \tag{A.26}$$

Physically this corresponds to bringing one of the NS5-branes to infinity that results to the two NS5-branes remaining with the semi-infinite branes attached to one of these NS5-branes (cf. (A.26) with (5.10)).

To conclude this Appendix, we see that there are, indeed, two different degenerations, one being characterized by the Lax operator (A.14) that contains two diagonal elements with the spectral parameter λ, and the other one whose Lax operator contains only one diagonal element with λ. Let us note that, unlike the $SL(2)$ case, bosonization of the both $SL(3)$ limiting degenerations looks quite involved.

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